

Affine reductive spaces of small dimension and left A-loops

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Abstract

In this paper we determine the at least 4-dimensional affine reductive homogeneous manifolds for an at most 9-dimensional simple Lie group or an at most 6-dimensional semi-simple Lie group. Those reductive spaces among them which admit a sharply transitive differentiable section yield local almost differentiable left A-loops. Using this we classify all global almost differentiable left A-loops L having either a 6-dimensional semi-simple Lie group or the group $SL_3(\mathbb{R})$ as the group topologically generated by their left translations. Moreover, we determine all at most 5-dimensional left A-loops L with $PSU_3(\mathbb{C}, 1)$ as the group topologically generated by their left translations.

1 Introduction

The affine reductive spaces are essential objects of differential geometry (cf. [8], [19], [12]). They are homogeneous manifolds G/H such that there exists an $Ad(H)$ -invariant subspace \mathfrak{m} of the Lie algebra \mathfrak{g} of G that is complementary to the subalgebra \mathfrak{h} in \mathfrak{g} .

The explicit knowledge of affine reductive spaces plays an important role in many investigations (cf. [21], [4], [13]). This paper is an application to differentiable loops since the affine reductive spaces are the key for the classification of almost differentiable left A-loops L ; these are loops in which any mapping $x \mapsto [(ab)^{-1}(a(bx))]$, $a, b \in L$ is an automorphism of L . The relations between them and reductive homogeneous spaces are explicitly discussed in [10], [11] and [18].

Using the fact that the groups topologically generated by the left translations of almost differentiable left A-loops L are Lie groups (cf. [17]), we treat L as images of global differentiable sections $\sigma : G/H \rightarrow G$, where G is a connected

Lie group, H is a closed subgroup containing no non-trivial normal subgroup of G such that the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H . Since the tangent space $T_1(\sigma(G/H))$ is a complementary reductive subspace to the Lie algebra \mathfrak{h} of H the affine reductive spaces are crucial for the classification of almost differentiable left A-loops.

In contrast to the compact connected Lie groups in which for any connected closed subgroup there is an reductive complement (cf. [12], p. 199), for non-compact Lie groups the situation is complicated already if they have small dimension. This is documented by Section 3 and Proposition 20, where we determine all at least 4-dimensional affine reductive homogeneous spaces $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m})$, such that \mathfrak{g} is either an at most 9-dimensional simple Lie algebra or it is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2$, where \mathfrak{g}_2 is a 3-dimensional simple Lie algebra.

The exponential images $\exp \mathfrak{m}$ of reductive complements \mathfrak{m} of the triples $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m})$ obtained in Section 3 and in Proposition 20 yield local left A-loops. In Section 4 and Proposition 21 we discuss which of these left A-loops can be extended to global ones. They are precisely those exponential images $\exp \mathfrak{m}$ which form systems of representatives for the cosets $\{xH \mid x \in G\}$ in G and do not contain any element conjugate to an element of H .

Since differentiable Bruck loops have realizations on differentiable affine symmetric spaces G/H , where H is the set of fixed elements of an involutory automorphism of G and $\sigma(G/H)$ is the exponential image of the (-1) -eigenspace of the corresponding automorphism of the Lie algebra \mathfrak{g} of G , the class of differentiable Bruck loops form a proper subclass of almost differentiable left A-loops. An important subclass of Bruck loops are the Bruck loops of hyperbolic type which correspond to Lie groups G and involutions τ fixing elementwise a maximal compact subgroup of G (cf. [7], 64.9, 64.10). Almost differentiable left A-loops L having dimension at most 3 and semi-simple Lie groups as the groups topologically generated by their left translations are classified in [18], Section 27 and in [6]. Hence in the following main result of this paper only at most 4-dimensional almost differentiable left A-loops occur.

Theorem *Let L be a connected almost differentiable left A-loop such that $\dim L \geq 4$ and the group topologically generated by the left translations of L is semi-simple.*

If $\dim G = 6$ then G is isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where G_2 is either $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$ and the loop L is either a Scheerer extension of G_2 by the hyperbolic plane loop \mathbb{H}_2 (cf. [18], Section 22) or the direct product $\mathbb{H}_2 \times \mathbb{H}_2$.

If the group G is simple and $7 \leq \dim G \leq 9$ then G is isomorphic either to $SL_3(\mathbb{R})$ or to $PSU_3(\mathbb{C}, 1)$. In the first case L is the 5-dimensional Bruck loop of hyperbolic type having the group $SO_3(\mathbb{R})$ as the stabilizer of $e \in L$ (cf. [5],

p. 12). In the case $G \cong PSU_3(\mathbb{C}, 1)$ every loop L with $\dim L < 6$ is the complex hyperbolic plane loop L_0 having the group $Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$ as the stabilizer of $e \in L_0$ (cf. [5], p. 9).

2 Some basic notions

A binary system (L, \cdot) is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b / a$. Let (L_1, \cdot) and $(L_2, *)$ be two loops. The set $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the componentwise multiplication is again a loop, which is called the direct product of L_1 and L_2 , and the loops (L_1, \cdot) , $(L_2, *)$ are subloops of L .

A loop is called a left A-loop if each mapping $\lambda_{x,y} = \lambda_{xy}^{-1} \lambda_x \lambda_y : L \rightarrow L$ is an automorphism of L .

Let G be the group generated by the left translations of L and let H be the stabilizer of $e \in L$ in the group G . The left translations of L form a subset of G acting on the cosets $\{xH; x \in G\}$ such that for any given cosets aH and bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.

Conversely, let G be a group, H be a subgroup containing no normal non-trivial subgroup of G and $\sigma : G/H \rightarrow G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets $\{xH, x \in G\}$ acts sharply transitively on the space G/H of $\{xH, x \in G\}$ (cf. [18], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space G/H or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group G is isomorphic to the group generated by the left translations of $L(\sigma)$.

If G is a Lie group and σ is a differentiable section satisfying the above conditions then the loop $L(\sigma)$ is almost differentiable. This loop is a left A-loop if and only if the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H . Moreover the manifold L is parallelizable since the set of the left translations is sharply transitive.

Let L_1 be a loop defined on the factor space G_1/H_1 with respect to a section $\sigma_1 : G_1/H_1 \rightarrow G_1$ the image of which is the set $M_1 \subset G_1$. Let G_2 be a group, let $\varphi : H_1 \rightarrow G_2$ be a homomorphism and $(H_1, \varphi(H_1)) = \{(x, \varphi(x)); x \in H_1\}$. A loop L is called a Scheerer extension of G_2 by L_1 if the loop L is defined on the factor space $(G_1 \times G_2)/(H_1, \varphi(H_1))$ with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \rightarrow G_1 \times G_2$ the image of which is the set $M_1 \times G_2$.

If L is a connected almost differentiable left A-loop, then the group G topologically generated by the left translations of L within the group of auto-homeomorphisms is a connected Lie group (cf. [17]; [18], Proposition 5.20).

p. 75), and we may describe L by a differentiable section.

Let L be a connected almost differentiable left A-loop. Let G be the Lie group topologically generated by the left translations of L , and let $(\mathbf{g}, [.,.])$ be the Lie algebra of G . Denote by \mathbf{h} the Lie algebra of the stabilizer H of the identity $e \in L$ in G and by $\mathbf{m} = T_1\sigma(G/H)$ the tangent space at $1 \in G$ of the image of the section $\sigma : G/H \rightarrow G$ corresponding to L . Then \mathbf{m} generates \mathbf{g} and the homogeneous space G/H is reductive, i.e. we have $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$ and $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. (cf. [18], Proposition 5.20. p. 75) If $[\mathbf{m}, \mathbf{m}] \subseteq \mathbf{h}$ then the factor space G/H is an affine symmetric space ([16]) and the corresponding loop L is called a Bruck loop.

In our computation we often use the following facts about the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$.

As a real basis of $\mathfrak{sl}_2(\mathbb{R})$ we choose the following

$$(*) \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(cf. [9], pp. 19-20).

With respect to this basis the Lie algebra multiplication is given by:

$$[e_1, e_2] = 2e_3, [e_1, e_3] = 2e_2, [e_3, e_2] = 2e_1.$$

1.1 An element $X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathfrak{sl}_2(\mathbb{R})$ is elliptic, parabolic or hyperbolic according whether

$$k(X) = k(X, X) = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \text{ is smaller, equal, or greater 0.}$$

The basis elements e_1, e_2 are hyperbolic, e_3 is elliptic and the elements $e_2 + e_3, e_1 + e_3$ are both parabolic. All elliptic elements, all hyperbolic elements as well as all parabolic elements of $\mathfrak{sl}_2(\mathbb{R})$ are conjugate in this order to e_3 , to e_1 respectively to $e_2 + e_3$ (cf. [9], p. 23). There are 3 conjugacy classes of the one dimensional subgroups of $PSL_2(\mathbb{R})$. As representatives of these classes we can choose $\exp e_3, \exp e_1, \exp e_2 + e_3$. There is precisely one conjugacy class \mathcal{C} of the two dimensional subgroups of $PSL_2(\mathbb{R})$, as a representative of \mathcal{C} we choose

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a > 0, b \in \mathbb{R} \right\}.$$

The Lie algebra of \mathcal{L}_2 is generated by the elements $e_1, e_2 + e_3$.

According to [9] for the exponential function $\exp : \mathfrak{sl}_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})$ we have

$$\exp X = C(k(X)) I + S(k(X)) X.$$

Here is

$$C(x) = \begin{cases} \cosh \sqrt{x} & \text{for } 0 \leq x, \\ \cos \sqrt{-x} & \text{for } 0 > x, \end{cases} \quad \sqrt{|x|} S(x) = \begin{cases} \sinh \sqrt{x} & \text{for } 0 \leq x, \\ \sin \sqrt{-x} & \text{for } 0 > x. \end{cases}$$

1.2 As a real basis of the Lie algebra $\mathfrak{so}_3(\mathbb{R}) \cong \mathfrak{su}_2(\mathbb{C})$ we can choose the basis elements $\{ie_1, ie_2, e_3\}$, where $i^2 = -1$. Every element of $\mathfrak{so}_3(\mathbb{R})$ is conjugate to e_3 .

If $X \in \mathfrak{so}_3(\mathbb{R})$ has the decomposition

$$X = \lambda_1 ie_1 + \lambda_2 ie_2 + \lambda_3 e_3$$

then the normalized real Cartan-Killing form $k : \mathfrak{so}_3(\mathbb{R}) \times \mathfrak{so}_3(\mathbb{R}) \rightarrow \mathbb{R}$; $k(X, Y) = \frac{1}{8} \text{trace}(\text{ad}X \text{ ad}Y)$ satisfies

$$k(X) = k(X, X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2.$$

For the exponential function $\exp : \mathfrak{su}_2(\mathbb{C}) \rightarrow SU_2(\mathbb{C})$ one has

$$\exp X = C(k(X)) I + S(k(X)) X,$$

where $C(x) = \cosh(\sqrt{-x}i)$ and $S(x) = \frac{\sinh(\sqrt{-x}i)}{\sqrt{-x}i}$.

Proposition 1. *There is no connected almost differentiable left A-loop L such that the group G topologically generated by its left translations is a compact quasi-simple Lie group G with $\dim G \leq 9$.*

Proof. If G is a quasi-simple Lie group then it admits a continuous section if and only if G is locally isomorphic to $SO_8(\mathbb{R})$ (cf. [20], pp. 149-150). \square

An important tool to exclude certain stabilizers H is the fundamental group π_1 of a connected topological space. This shows the following lemma which is proved in [5], p. 6.

Lemma 2. *Denote by G a connected Lie group and by H a connected subgroup of G . Let $\sigma : G/H \rightarrow G$ be a global section. Then $\pi_1(K) \cong \pi_1(\sigma(G/H)) \times \pi_1(K_1)$, where K respectively K_1 is a maximal compact subgroup of G respectively H .*

From [6] we use Lemma 2, which reads as follows.

Lemma 3. *Let L be an almost differentiable loop and denote by \mathfrak{m} the tangent space $T_1\sigma(G/H)$, where $\sigma : G/H \rightarrow G$ is the section corresponding to L . Then \mathfrak{m} does not contain any element of $\text{Ad}_g \mathfrak{h}$ for some $g \in G$. Moreover, every element of G can be uniquely written as a product of an element of $\sigma(G/H)$ with an element of H .*

3 Affine reductive spaces of small dimension

In this section we determine all affine reductive homogeneous spaces $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m})$, where \mathfrak{g} is a simple non-compact Lie algebra of dimension at most 9 and \mathfrak{h} is a subalgebra of \mathfrak{g} such that $\dim \mathfrak{g} - \dim \mathfrak{h} > 3$.

First we deal with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. A real basis of \mathfrak{g} is given by $\{e_1, e_2, e_3, ie_1, ie_2, ie_3\}$, where $\{e_1, e_2, e_3\}$ is the basis of $\mathfrak{sl}_2(\mathbb{R})$ described by (*).

Using the classification of Lie (see Theorem 15 in [15], p. 129) we obtain that every 2-dimensional Lie algebra \mathfrak{h} of \mathfrak{g} has (up to conjugation) one of the following shapes:

$$\mathfrak{h}_1 = \langle e_1, e_2 + e_3 \rangle, \quad \mathfrak{h}_2 = \langle i(e_2 + e_3), e_2 + e_3 \rangle, \quad \mathfrak{h}_3 = \langle e_3, ie_3 \rangle,$$

and every 1-dimensional Lie algebra \mathfrak{h} of \mathfrak{g} is one of the following:

$$\mathfrak{h}_4 = \langle e_1 \rangle, \quad \mathfrak{h}_5 = \langle e_2 + e_3 \rangle, \quad \mathfrak{h}_6 = \langle e_3 \rangle.$$

Proposition 4. *The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is reductive with respect to the following pairs $(\mathfrak{h}, \mathfrak{m})$, where \mathfrak{h} is an at most 2-dimensional subalgebra of \mathfrak{g} and \mathfrak{m} is a complementary subspace to \mathfrak{h} generating \mathfrak{g}*

- 1) $\mathfrak{h}_3 = \langle e_3, ie_3 \rangle$, $\mathfrak{m} = \langle e_1, e_2, ie_1, ie_2 \rangle$,
- 2) $\mathfrak{h}_4 = \langle e_1 \rangle$, $\mathfrak{m}_a = \langle e_2, e_3, ie_1 + ae_1, ie_2, ie_3 \rangle$, where $a \in \mathbb{R}$,
- 3) $\mathfrak{h}_6 = \langle e_3 \rangle$, $\mathfrak{m}_b = \langle e_1, e_2, ie_1, ie_2, ie_3 + be_3 \rangle$, where $b \in \mathbb{R}$.

Proof. The basis elements of an arbitrary complement \mathfrak{m}_1 to \mathfrak{h}_1 in \mathfrak{g} are

$$\begin{aligned} X_1 &= e_2 + a_1 e_1 + b_1(e_2 + e_3), & X_2 &= ie_1 + a_2 e_1 + b_2(e_2 + e_3), \\ X_3 &= ie_2 + a_3 e_1 + b_3(e_2 + e_3), & X_4 &= ie_3 + a_4 e_1 + b_4(e_2 + e_3), \end{aligned}$$

where $a_j, b_j, j = 1, 2, 3, 4$ are real parameters.

An arbitrary complement \mathfrak{m}_2 to \mathfrak{h}_2 in \mathfrak{g} has as generators

$$\begin{aligned} Y_1 &= e_1 + a_1(e_2 + e_3) + b_1 i(e_2 + e_3), & Y_2 &= e_2 + a_2(e_2 + e_3) + b_2 i(e_2 + e_3), \\ Y_3 &= ie_1 + a_3(e_2 + e_3) + b_3 i(e_2 + e_3), & Y_4 &= ie_2 + a_4(e_2 + e_3) + b_4 i(e_2 + e_3), \end{aligned}$$

where $a_j, b_j \in \mathbb{R}, j = 1, 2, 3, 4$.

We can choose as basis elements of an arbitrary complement \mathfrak{m}_3 to \mathfrak{h}_3 the following:

$$\begin{aligned} Z_1 &= e_1 + a_1 e_3 + b_1 ie_3, & Z_2 &= e_2 + a_2 e_3 + b_2 ie_3, \\ Z_3 &= ie_1 + a_3 e_3 + b_3 ie_3, & Z_4 &= ie_2 + a_4 e_3 + b_4 ie_3, \end{aligned}$$

where $a_j, b_j \in \mathbb{R}, j = 1, 2, 3, 4$ are real numbers.

An arbitrary complement \mathfrak{m}_4 to \mathfrak{h}_4 in \mathfrak{g} has as basis elements

$$\begin{aligned} W_1 &= e_2 + a_1 e_1, & W_2 &= e_3 + a_2 e_1, & W_3 &= ie_1 + a_3 e_1, \\ W_4 &= ie_2 + a_4 e_1, & W_5 &= ie_3 + a_5 e_1 \end{aligned}$$

with the real parameters $a_j, j = 1, 2, 3, 4, 5$.

The generators of an arbitrary complement \mathfrak{m}_5 to \mathfrak{h}_5 in \mathfrak{g} are

$$V_1 = e_1 + a_1(e_2 + e_3), \quad V_2 = e_2 + a_2(e_2 + e_3), \quad V_3 = ie_1 + a_3(e_2 + e_3),$$

$$V_4 = ie_2 + a_4(e_2 + e_3), \quad V_5 = ie_3 + a_5(e_2 + e_3),$$

where $a_j, j = 1, 2, 3, 4, 5$ are real parameters.

An arbitrary complement \mathbf{m}_6 to \mathbf{h}_6 in \mathbf{g} has as generators

$$U_1 = e_1 + a_1e_3, \quad U_2 = e_2 + a_2e_3, \quad U_3 = ie_1 + a_3e_3, \\ U_4 = ie_2 + a_4e_3, \quad U_5 = ie_3 + a_5e_3$$

with $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$.

Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i, i = 1, \dots, 6$, we obtain the contradictions that $[e_2 + e_3, X_1] = 2e_1 - 2a_1(e_2 + e_3) \in \mathbf{h}_1$ and $[e_2 + e_3, Y_1] = [e_2 + e_3, V_1] = -2(e_2 + e_3) \in \mathbf{h}_2 \cap \mathbf{h}_5$ and the assertion follows. \square

Now we consider the Lie algebra $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$. It is isomorphic to the Lie algebra of matrices

$$(\lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3 + \lambda_4e_4 + \lambda_5e_5 + \lambda_6e_6 + \lambda_7e_7 + \lambda_8e_8) \mapsto \\ \begin{pmatrix} -\lambda_5 - \lambda_8 & \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_5 & \lambda_6 \\ \lambda_4 & \lambda_7 & \lambda_8 \end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \dots, 8.$$

In this representation the Lie multiplication of \mathbf{g} is given by

$$[e_1, e_2] = [e_1, e_7] = [e_2, e_6] = [e_3, e_4] = [e_3, e_6] = [e_4, e_7] = [e_5, e_8] = 0, \\ [e_1, e_6] = [e_2, e_5] = \frac{1}{2}[e_2, e_8] = e_2, \quad [e_1, e_8] = [e_2, e_7] = \frac{1}{2}[e_1, e_5] = e_1, \\ [e_4, e_6] = [e_3, e_8] = \frac{1}{2}[e_3, e_5] = -e_3, \quad [e_3, e_7] = [e_4, e_5] = \frac{1}{2}[e_4, e_8] = -e_4, \\ [e_6, e_8] = [e_5, e_6] = [e_3, e_2] = e_6, \quad [e_1, e_4] = [e_5, e_7] = [e_7, e_8] = -e_7, \\ [e_1, e_3] = -e_5, \quad [e_2, e_4] = -e_8, \quad [e_6, e_7] = e_5 - e_8.$$

Now using the classification of Lie, who has determined all subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ (cf. [15], pp. 288-289 and [14], p. 384) we obtain that every 4-dimensional Lie algebra \mathbf{h} of \mathbf{g} has (up to conjugation) one of the following forms:

$$\mathbf{h}_1 = \langle e_1, e_2, e_6, e_5 + ce_8 \rangle, \quad \mathbf{h}_2 = \langle e_3, e_5, e_6, e_8 \rangle, \quad \mathbf{h}_3 = \langle e_1, e_2, e_6, e_8 \rangle, \\ \mathbf{h}_4 = \langle e_2, e_5, e_6, e_8 \rangle, \quad \mathbf{h}_5 \cong \mathfrak{gl}_2(\mathbb{R}) = \langle e_5, e_6, e_7, e_8 \rangle, \text{ where } c \in \mathbb{R}.$$

The 3-dimensional subalgebras \mathbf{h} of \mathbf{g} (up to conjugation) are the following:

$$\mathbf{h}_6 \cong \mathfrak{so}_3(\mathbb{R}) = \langle e_1 - e_3, e_2 - e_4, e_7 - e_6 \rangle, \quad \mathbf{h}_7 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_1 + e_3, e_2 + e_4, e_6 - e_7 \rangle, \\ \mathbf{h}_8 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_5 - e_8, e_6, e_7 \rangle, \quad \mathbf{h}_9 = \langle a(e_5 + e_8) + e_6 - e_7, e_1, e_2 \rangle, \quad a \geq 0, \\ \mathbf{h}_{10} = \langle e_5 - e_8, e_2 + e_3, e_6 \rangle, \quad \mathbf{h}_{11} = \langle e_3, e_6, e_8 + e_2 \rangle, \quad \mathbf{h}_{12} = \langle e_2, e_6, e_5 + e_8 - e_3 \rangle, \\ \mathbf{h}_{13} = \langle e_1, e_2, e_6 \rangle, \quad \mathbf{h}_{14} = \langle e_5, e_8, e_6 \rangle, \quad \mathbf{h}_{15} = \langle e_2, e_5 + e_8, e_6 \rangle, \quad \mathbf{h}_{16} = \langle e_3, e_6, e_8 \rangle, \\ \mathbf{h}_{17} = \langle e_2, e_6, (b-1)e_5 + be_8 \rangle, \quad b \in \mathbb{R}, \quad \mathbf{h}_{18} = \langle e_3, e_6, e_5 + ce_8 \rangle, \quad c \in \mathbb{R}.$$

The 2-dimensional subalgebras \mathbf{h} of \mathbf{g} are given (up to conjugation) by

$$\begin{aligned} \mathbf{h}_{19} &= \langle e_6, e_2 + e_3 \rangle, & \mathbf{h}_{20} &= \langle e_6, e_2 + e_8 \rangle, & \mathbf{h}_{21} &= \langle e_3, e_6 + e_5 \rangle, \\ \mathbf{h}_{22} &= \langle e_3, e_5 + ae_8 \rangle, \quad a \in \mathbb{R} \setminus \{0, 1\}, & \mathbf{h}_{23} &= \langle e_5, e_6 \rangle, & \mathbf{h}_{24} &= \langle e_2, e_6 \rangle, \\ \mathbf{h}_{25} &= \langle e_6, e_3 \rangle, & \mathbf{h}_{26} &= \langle e_5, e_8 \rangle, & \mathbf{h}_{27} &= \langle e_6, e_5 + e_8 \rangle, & \mathbf{h}_{28} &= \langle e_6, e_8 \rangle, \\ \mathbf{h}_{29} &= \langle e_5 - e_8, e_2 + e_3 \rangle, & \mathbf{h}_{30} &= \langle e_5 + e_8, e_6 - e_7 \rangle. \end{aligned}$$

Moreover, every 1-dimensional subalgebra \mathbf{h} of \mathbf{g} has one of the following shapes:

$$\begin{aligned} \mathbf{h}_{31} &= \langle e_5 + ae_8 \rangle, \quad a \in \mathbb{R} \setminus \{0\}, & \mathbf{h}_{32} &= \langle e_2 + e_8 \rangle, & \mathbf{h}_{33} &= \langle e_2 + e_3 \rangle, \\ \mathbf{h}_{34} &= \langle e_6 \rangle, & \mathbf{h}_{35} &= \langle e_6 - e_7 + b(e_5 + e_8) \rangle, \quad b \geq 0. \end{aligned}$$

Proposition 5. *The Lie algebra $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ is reductive with respect to a 4-dimensional subalgebra \mathbf{h} of \mathbf{g} and a complementary subspace \mathbf{m} generating \mathbf{g} only in the case $\mathbf{h}_5 \cong \mathfrak{gl}_2(\mathbb{R})$ and $\mathbf{m}_5 = \langle e_1, e_2, e_3, e_4 \rangle$.*

Proof. The basis elements of an arbitrary complement \mathbf{m}_i to the subalgebra \mathbf{h}_i are:

For $i = 1$

$$\begin{aligned} &e_3 + a_1e_1 + a_2e_2 + a_3(e_5 + ce_8) + a_4e_6, \quad e_4 + b_1e_1 + b_2e_2 + b_3(e_5 + ce_8) + b_4e_6, \\ &e_7 + c_1e_1 + c_2e_2 + c_3(e_5 + ce_8) + c_4e_6, \quad e_8 + d_1e_1 + d_2e_2 + d_3(e_5 + ce_8) + d_4e_6, \end{aligned}$$

for $i = 2$

$$\begin{aligned} &e_1 + a_1e_3 + a_2e_5 + a_3e_6 + a_4e_8, \quad e_2 + b_1e_3 + b_2e_5 + b_3e_6 + b_4e_8, \\ &e_4 + c_1e_3 + c_2e_5 + c_3e_6 + c_4e_8, \quad e_7 + d_1e_3 + d_2e_5 + d_3e_6 + d_4e_8, \end{aligned}$$

for $i = 3$

$$\begin{aligned} &e_3 + a_1e_1 + a_2e_2 + a_3e_6 + a_4e_8, \quad e_4 + b_1e_1 + b_2e_2 + b_3e_6 + b_4e_8, \\ &e_5 + c_1e_1 + c_2e_2 + c_3e_6 + c_4e_8, \quad e_7 + d_1e_1 + d_2e_2 + d_3e_6 + d_4e_8, \end{aligned}$$

for $i = 4$

$$\begin{aligned} &e_1 + a_1e_2 + a_2e_5 + a_3e_6 + a_4e_8, \quad e_3 + b_1e_2 + b_2e_5 + b_3e_6 + b_4e_8, \\ &e_4 + c_1e_2 + c_2e_5 + c_3e_6 + c_4e_8, \quad e_7 + d_1e_2 + d_2e_5 + d_3e_6 + d_4e_8, \end{aligned}$$

for $i = 5$

$$\begin{aligned} &e_1 + a_1e_5 + a_2e_6 + a_3e_7 + a_4e_8, \quad e_2 + b_1e_5 + b_2e_6 + b_3e_7 + b_4e_8, \\ &e_3 + c_1e_5 + c_2e_6 + c_3e_7 + c_4e_8, \quad e_4 + d_1e_5 + d_2e_6 + d_3e_7 + d_4e_8, \end{aligned}$$

where a_j, b_j, c_j, d_j are real numbers $j = 1, 2, 3, 4$. The assertion follows now from the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. \square

Proposition 6. *The Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ is reductive with a 3-dimensional subalgebra \mathfrak{h} and a 5-dimensional complementary subspace \mathfrak{m} generating \mathfrak{g} in precisely one of the following cases:*

- 1) $\mathfrak{h}_6 \cong \mathfrak{so}_3(\mathbb{R})$, $\mathfrak{m}_6 = \langle e_5, e_8, e_1 + e_3, e_2 + e_4, e_7 + e_6 \rangle$,
- 2) $\mathfrak{h}_7 = \langle e_1 + e_3, e_2 + e_4, e_6 - e_7 \rangle$, $\mathfrak{m}_7 = \langle e_5, e_8, e_1 - e_3, e_2 - e_4, e_7 + e_6 \rangle$,
- 3) $\mathfrak{h}_8 = \langle e_5 - e_8, e_6, e_7 \rangle$, $\mathfrak{m}_8 = \langle e_1, e_2, e_3, e_4, e_5 + e_8 \rangle$.

Both Lie algebras \mathfrak{h}_7 and \mathfrak{h}_8 are isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

Proof. The generators of an arbitrary complement \mathfrak{m}_i to \mathfrak{h}_i in \mathfrak{g} are:
For $i = 6$

$$\begin{aligned} &e_3 + a_1(e_1 - e_3) + a_2(e_2 - e_4) + a_3(e_7 - e_6), \\ &e_4 + b_1(e_1 - e_3) + b_2(e_2 - e_4) + b_3(e_7 - e_6), \\ &e_5 + c_1(e_1 - e_3) + c_2(e_2 - e_4) + c_3(e_7 - e_6), \\ &e_6 + d_1(e_1 - e_3) + d_2(e_2 - e_4) + d_3(e_7 - e_6), \\ &e_8 + f_1(e_1 - e_3) + f_2(e_2 - e_4) + f_3(e_7 - e_6), \end{aligned}$$

for $i = 7$

$$\begin{aligned} &e_3 + a_1(e_1 + e_3) + a_2(e_2 + e_4) + a_3(e_6 - e_7), \\ &e_4 + b_1(e_1 + e_3) + b_2(e_2 + e_4) + b_3(e_6 - e_7), \\ &e_5 + c_1(e_1 + e_3) + c_2(e_2 + e_4) + c_3(e_6 - e_7), \\ &e_6 + d_1(e_1 + e_3) + d_2(e_2 + e_4) + d_3(e_6 - e_7), \\ &e_8 + f_1(e_1 + e_3) + f_2(e_2 + e_4) + f_3(e_6 - e_7), \end{aligned}$$

for $i = 8$

$$\begin{aligned} &e_1 + a_1(e_5 - e_8) + a_2e_6 + a_3e_7, \quad e_2 + b_1(e_5 - e_8) + b_2e_6 + b_3e_7, \\ &e_3 + c_1(e_5 - e_8) + c_2e_6 + c_3e_7, \quad e_4 + d_1(e_5 - e_8) + d_2e_6 + d_3e_7, \\ &e_5 + f_1(e_5 - e_8) + f_2e_6 + f_3e_7, \end{aligned}$$

for $i = 9$

$$\begin{aligned} &e_3 + a_1e_1 + a_2e_2 + a_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_4 + b_1e_1 + b_2e_2 + b_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_5 + c_1e_1 + c_2e_2 + c_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_6 + d_1e_1 + d_2e_2 + d_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_8 + f_1e_1 + f_2e_2 + f_3(e_6 - e_7 + a(e_5 + e_8)), \end{aligned}$$

for $i = 10$

$$e_1 + a_1(e_2 + e_3) + a_2(e_5 - e_8) + a_3e_6, \quad e_2 + b_1(e_2 + e_3) + b_2(e_5 - e_8) + b_3e_6,$$

$$e_4 + c_1(e_2 + e_3) + c_2(e_5 - e_8) + c_3e_6, \quad e_5 + d_1(e_2 + e_3) + d_2(e_5 - e_8) + d_3e_6, \\ e_7 + f_1(e_2 + e_3) + f_2(e_5 - e_8) + f_3e_6,$$

for $i = 11$

$$e_1 + a_1(e_2 + e_8) + a_2e_3 + a_3e_6, \quad e_2 + b_1(e_2 + e_8) + b_2e_3 + b_3e_6, \\ e_4 + c_1(e_2 + e_8) + c_2e_3 + c_3e_6, \quad e_5 + d_1(e_2 + e_8) + d_2e_3 + d_3e_6, \\ e_7 + f_1(e_2 + e_8) + f_2e_3 + f_3e_6,$$

for $i = 12$

$$e_1 + a_1e_2 + a_2e_6 + a_3(e_5 + e_8 - e_3), \quad e_3 + b_1e_2 + b_2e_6 + b_3(e_5 + e_8 - e_3), \\ e_4 + c_1e_2 + c_2e_6 + c_3(e_5 + e_8 - e_3), \quad e_7 + d_1e_2 + d_2e_6 + d_3(e_5 + e_8 - e_3), \\ e_8 + f_1e_2 + f_2e_6 + f_3(e_5 + e_8 - e_3),$$

for $i = 13$

$$e_3 + a_1e_1 + a_2e_2 + a_3e_6, \quad e_4 + b_1e_1 + b_2e_2 + b_3e_6, \quad e_5 + c_1e_1 + c_2e_2 + c_3e_6, \\ e_7 + d_1e_1 + d_2e_2 + d_3e_6, \quad e_8 + f_1e_1 + f_2e_2 + f_3e_6,$$

for $i = 14$

$$e_1 + a_1e_5 + a_2e_6 + a_3e_8, \quad e_2 + b_1e_5 + b_2e_6 + b_3e_8, \quad e_3 + c_1e_5 + c_2e_6 + c_3e_8, \\ e_4 + d_1e_5 + d_2e_6 + d_3e_8, \quad e_7 + f_1e_5 + f_2e_6 + f_3e_8,$$

for $i = 15$

$$e_1 + a_1e_2 + a_2(e_5 + e_8) + a_3e_6, \quad e_3 + b_1e_2 + b_2(e_5 + e_8) + b_3e_6, \\ e_4 + c_1e_2 + c_2(e_5 + e_8) + c_3e_6, \quad e_5 + d_1e_2 + d_2(e_5 + e_8) + d_3e_6, \\ e_7 + f_1e_2 + f_2(e_5 + e_8) + f_3e_6,$$

for $i = 16$

$$e_1 + a_1e_3 + a_2e_6 + a_3e_8, \quad e_2 + b_1e_3 + b_2e_6 + b_3e_8, \quad e_4 + c_1e_3 + c_2e_6 + c_3e_8, \\ e_5 + d_1e_3 + d_2e_6 + d_3e_8, \quad e_7 + f_1e_3 + f_2e_6 + f_3e_8,$$

for $i = 17$ and $b \neq 0$

$$e_1 + a_1e_2 + a_2e_6 + a_3((b-1)e_5 + be_8), \quad e_3 + b_1e_2 + b_2e_6 + b_3((b-1)e_5 + be_8), \\ e_4 + c_1e_2 + c_2e_6 + c_3((b-1)e_5 + be_8), \quad e_5 + d_1e_2 + d_2e_6 + d_3((b-1)e_5 + be_8), \\ e_7 + f_1e_2 + f_2e_6 + f_3((b-1)e_5 + be_8),$$

for $i = 17$ and $b = 0$

$$e_1 + a_1e_2 + a_2e_6 - a_3e_5, \quad e_3 + b_1e_2 + b_2e_6 - b_3e_5, \quad e_4 + c_1e_2 + c_2e_6 - c_3e_5, \\ e_7 + d_1e_2 + d_2e_6 - d_3e_5, \quad e_8 + f_1e_2 + f_2e_6 - f_3e_5,$$

for $i = 18$

$$\begin{aligned}
& e_1 + a_1 e_3 + a_2(e_5 + ce_8) + a_3 e_6, \quad e_2 + b_1 e_3 + b_2(e_5 + ce_8) + b_3 e_6, \\
& e_4 + c_1 e_3 + c_2(e_5 + ce_8) + c_3 e_6, \quad e_7 + d_1 e_3 + d_2(e_5 + ce_8) + d_3 e_6, \\
& e_8 + f_1 e_3 + f_2(e_5 + ce_8) + f_3 e_6,
\end{aligned}$$

where $a_j, b_j, c_j, d_j, f_j \in \mathbb{R}$, $j = 1, 2, 3$. Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 6, \dots, 18$, we obtain the assertion. \square

Proposition 7. *The Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ is reductive with respect to a pair (\mathbf{h}, \mathbf{m}) , where \mathbf{h} is a 2-dimensional subalgebra of \mathfrak{g} and \mathbf{m} is a complementary subspace to \mathbf{h} generating \mathfrak{g} in exactly one of the following cases:*

- 1) $\mathbf{h}_{26} = \langle e_5, e_8 \rangle$ and $\mathbf{m}_{26} = \langle e_1, e_2, e_3, e_4, e_6, e_7 \rangle$.
- 2) $\mathbf{h}_{30} = \langle e_5 + e_8, e_6 - e_7 \rangle$ and $\mathbf{m}_{30} = \langle e_1, e_2, e_3, e_4, e_5 - e_8, e_6 + e_7 \rangle$.

Proof. An arbitrary complement \mathbf{m}_i to the subalgebra \mathbf{h}_i , $i = 19, \dots, 30$, in \mathfrak{g} has as generators in the case $i = 19$

$$\begin{aligned}
& e_1 + b_1 e_6 + c_1(e_2 + e_3), \quad e_2 + b_2 e_6 + c_2(e_2 + e_3), \quad e_4 + b_3 e_6 + c_3(e_2 + e_3) \\
& e_5 + b_4 e_6 + c_4(e_2 + e_3), \quad e_7 + b_5 e_6 + c_5(e_2 + e_3), \quad e_8 + b_6 e_6 + c_6(e_2 + e_3),
\end{aligned}$$

in the case $i = 20$

$$\begin{aligned}
& e_1 + b_1 e_6 + c_1(e_2 + e_8), \quad e_2 + b_2 e_6 + c_2(e_2 + e_8), \quad e_3 + b_3 e_6 + c_3(e_2 + e_8), \\
& e_4 + b_4 e_6 + c_4(e_2 + e_8), \quad e_5 + b_5 e_6 + c_5(e_2 + e_8), \quad e_7 + b_6 e_6 + c_6(e_2 + e_8),
\end{aligned}$$

in the case $i = 21$

$$\begin{aligned}
& e_1 + b_1 e_3 + c_1(e_6 + e_5), \quad e_2 + b_2 e_3 + c_2(e_6 + e_5), \quad e_4 + b_3 e_3 + c_3(e_6 + e_5), \\
& e_5 + b_4 e_3 + c_4(e_6 + e_5), \quad e_7 + b_5 e_3 + c_5(e_6 + e_5), \quad e_8 + b_6 e_3 + c_6(e_6 + e_5),
\end{aligned}$$

in the case $i = 22$

$$\begin{aligned}
& e_1 + b_1 e_3 + c_1(e_5 + ae_8), \quad e_2 + b_2 e_3 + c_2(e_5 + ae_8), \quad e_4 + b_3 e_3 + c_3(e_5 + ae_8), \\
& e_6 + b_4 e_3 + c_4(e_5 + ae_8), \quad e_7 + b_5 e_3 + c_5(e_5 + ae_8), \quad e_8 + b_6 e_3 + c_6(e_5 + ae_8),
\end{aligned}$$

in the case $i = 23$

$$\begin{aligned}
& e_1 + b_1 e_5 + c_1 e_6, \quad e_2 + b_2 e_5 + c_2 e_6, \quad e_3 + b_3 e_5 + c_3 e_6, \\
& e_4 + b_4 e_5 + c_4 e_6, \quad e_7 + b_5 e_5 + c_5 e_6, \quad e_8 + b_6 e_5 + c_6 e_6,
\end{aligned}$$

in the case $i = 24$

$$\begin{aligned}
& e_1 + b_1 e_2 + c_1 e_6, \quad e_3 + b_2 e_2 + c_2 e_6, \quad e_4 + b_3 e_2 + c_3 e_6, \\
& e_5 + b_4 e_2 + c_4 e_6, \quad e_7 + b_5 e_2 + c_5 e_6, \quad e_8 + b_6 e_2 + c_6 e_6,
\end{aligned}$$

in the case $i = 25$

$$\begin{aligned}
& e_1 + b_1 e_3 + c_1 e_6, \quad e_2 + b_2 e_3 + c_2 e_6, \quad e_4 + b_3 e_3 + c_3 e_6, \\
& e_5 + b_4 e_3 + c_4 e_6, \quad e_7 + b_5 e_3 + c_5 e_6, \quad e_8 + b_6 e_3 + c_6 e_6,
\end{aligned}$$

in the case $i = 26$

$$\begin{aligned} e_1 + b_1 e_5 + c_1 e_8, \quad e_2 + b_2 e_5 + c_2 e_8, \quad e_3 + b_3 e_5 + c_3 e_8, \\ e_4 + b_4 e_5 + c_4 e_8, \quad e_6 + b_5 e_5 + c_5 e_8, \quad e_7 + b_6 e_5 + c_6 e_8, \end{aligned}$$

in the case $i = 27$

$$\begin{aligned} e_1 + b_1 e_6 + c_1(e_5 + e_8), \quad e_2 + b_2 e_6 + c_2(e_5 + e_8), \quad e_3 + b_3 e_6 + c_3(e_5 + e_8), \\ e_4 + b_4 e_6 + c_4(e_5 + e_8), \quad e_5 + b_5 e_6 + c_5(e_5 + e_8), \quad e_7 + b_6 e_6 + c_6(e_5 + e_8), \end{aligned}$$

in the case $i = 28$

$$\begin{aligned} e_1 + b_1 e_6 + c_1 e_8, \quad e_2 + b_2 e_6 + c_2 e_8, \quad e_3 + b_3 e_6 + c_3 e_8, \\ e_4 + b_4 e_6 + c_4 e_8, \quad e_5 + b_5 e_6 + c_5 e_8, \quad e_7 + b_6 e_6 + c_6 e_8, \end{aligned}$$

in the case $i = 29$

$$\begin{aligned} e_1 + b_1(e_2 + e_3) + c_1(e_5 - e_8), \quad e_2 + b_2(e_2 + e_3) + c_2(e_5 - e_8), \\ e_4 + b_3(e_2 + e_3) + c_3(e_5 - e_8), \quad e_5 + b_4(e_2 + e_3) + c_4(e_5 - e_8), \\ e_6 + b_5(e_2 + e_3) + c_5(e_5 - e_8), \quad e_7 + b_6(e_2 + e_3) + c_6(e_5 - e_8), \end{aligned}$$

in the case $i = 30$

$$\begin{aligned} e_1 + b_1(e_5 + e_8) + c_1(e_6 - e_7), \quad e_2 + b_2(e_5 + e_8) + c_2(e_6 - e_7), \\ e_3 + b_3(e_5 + e_8) + c_3(e_6 - e_7), \quad e_4 + b_4(e_5 + e_8) + c_4(e_6 - e_7), \\ e_5 + b_5(e_5 + e_8) + c_5(e_6 - e_7), \quad e_6 + b_6(e_5 + e_8) + c_6(e_6 - e_7), \end{aligned}$$

where $b_j, c_j \in \mathbb{R}$, $j = 1, \dots, 6$. The relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 19, \dots, 30$, yields the assertion. \square

Proposition 8. *The Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ is reductive with a 1-dimensional subalgebra \mathbf{h} and a 7-dimensional complementary subspace \mathbf{m} generating \mathfrak{g} in precisely one of the following cases:*

- 1) $\mathbf{h}_{31,1} = \langle e_5 + a e_8 \rangle$, $a \in \mathbb{R} \setminus \{0, 1, -\frac{1}{2}, -2\}$ and $\mathbf{m}_b = \langle e_1, e_2, e_3, e_4, e_6, e_7, e_8 + b(e_5 + a e_8) \rangle$, $b \in \mathbb{R}$,
- 2) $\mathbf{h}_{31,2} = \langle e_5 - 2e_8 \rangle$ and $\mathbf{m}_{b,c,d} = \langle e_6, e_7, e_1 + b(e_5 - 2e_8), e_3 + c(e_5 - 2e_8), e_2, e_4, e_8 + d(e_5 - 2e_8) \rangle$, $b, c, d \in \mathbb{R}$,
- 3) $\mathbf{h}_{31,3} = \langle e_5 - \frac{1}{2}e_8 \rangle$ and $\mathbf{m}_{b,c,d} = \langle e_6, e_7, e_1, e_2 + b(e_5 - \frac{1}{2}e_8), e_3, e_4 + c(e_5 - \frac{1}{2}e_8), e_8 + d(e_5 - \frac{1}{2}e_8) \rangle$, $b, c, d \in \mathbb{R}$,
- 4) $\mathbf{h}_{31,4} = \langle e_5 + e_8 \rangle$ and $\mathbf{m}_{b,c,d} = \langle e_1, e_2, e_3, e_4, e_6 + b(e_5 + e_8), e_7 + c(e_5 + e_8), e_8 + d(e_5 + e_8) \rangle$, $b, c, d \in \mathbb{R}$,
- 5) $\mathbf{h}_{32} = \langle e_2 + e_8 \rangle$ and $\mathbf{m}_d = \langle e_1, e_2, e_3, -e_8 + 2e_4, e_6, e_7, e_5 + d e_8 \rangle$, $d \in \mathbb{R}$,
- 6) $\mathbf{h}_{35} = \langle e_6 - e_7 + b(e_5 + e_8) \rangle$, $b \geq 0$ and $\mathbf{m}_c = \langle e_1, e_2, e_3, e_4, e_6 + e_7, e_5 - e_8, e_8 - 2c e_7 + 2c b e_8 \rangle$, $c \in \mathbb{R}$.

Proof. An arbitrary complement \mathbf{m}_i to the subalgebra \mathbf{h}_i , $i = 31, \dots, 35$, in \mathbf{g} has as generators in the case $i = 31$

$$e_1 + a_1(e_5 + ae_8), \quad e_2 + a_2(e_5 + ae_8), \quad e_3 + a_3(e_5 + ae_8), \quad e_4 + a_4(e_5 + ae_8), \\ e_6 + a_5(e_5 + ae_8), \quad e_7 + a_6(e_5 + ae_8), \quad e_8 + a_7(e_5 + ae_8),$$

in the case $i = 32$

$$e_1 + a_1(e_2 + e_8), \quad e_3 + a_2(e_2 + e_8), \quad e_4 + a_3(e_2 + e_8), \quad e_5 + a_4(e_2 + e_8), \\ e_6 + a_5(e_2 + e_8), \quad e_7 + a_6(e_2 + e_8), \quad e_8 + a_7(e_2 + e_8),$$

in the case $i = 33$

$$e_1 + a_1(e_2 + e_3), \quad e_3 + a_2(e_2 + e_3), \quad e_4 + a_3(e_2 + e_3), \quad e_5 + a_4(e_2 + e_3), \\ e_6 + a_5(e_2 + e_3), \quad e_7 + a_6(e_2 + e_3), \quad e_8 + a_7(e_2 + e_3),$$

in the case $i = 34$

$$e_1 + a_1e_6, \quad e_2 + a_2e_6, \quad e_3 + a_3e_6, \quad e_4 + a_4e_6, \quad e_5 + a_5e_6, \\ e_7 + a_6e_6, \quad e_8 + a_7e_6,$$

in the case $i = 35$

$$e_1 + a_1(e_6 - e_7 + b(e_5 + e_8)), \quad e_2 + a_2(e_6 - e_7 + b(e_5 + e_8)), \\ e_3 + a_3(e_6 - e_7 + b(e_5 + e_8)), \quad e_4 + a_4(e_6 - e_7 + b(e_5 + e_8)), \\ e_5 + a_5(e_6 - e_7 + b(e_5 + e_8)), \quad e_7 + a_6(e_6 - e_7 + b(e_5 + e_8)), \\ e_8 + a_7(e_6 - e_7 + b(e_5 + e_8)),$$

where $a_j \in \mathbb{R}$, $j = 1, \dots, 7$. Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 31, \dots, 35$, we obtain the assertion. \square

Now we deal with the Lie algebra $\mathfrak{su}_3(\mathbb{C}, 1)$. It can be treated as the Lie algebra of matrices

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto \\ \begin{pmatrix} -\lambda_1 i & -\lambda_2 - \lambda_3 i & \lambda_4 + \lambda_5 i \\ \lambda_2 - \lambda_3 i & \lambda_1 i + \lambda_6 i & \lambda_7 + \lambda_8 i \\ \lambda_4 - \lambda_5 i & \lambda_7 - \lambda_8 i & -\lambda_6 i \end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \dots, 8.$$

Then the multiplication of \mathbf{g} is given by the following:

$$[e_1, e_6] = 0, \quad [e_3, e_2] = 2e_1, \quad [e_4, e_5] = 2(e_1 - e_6), \quad [e_8, e_7] = 2e_6, \\ [e_6, e_3] = [e_7, e_4] = [e_8, e_5] = \frac{1}{2}[e_1, e_3] = e_2, \\ [e_2, e_6] = [e_4, e_8] = [e_7, e_5] = \frac{1}{2}[e_2, e_1] = e_3,$$

$$\begin{aligned}
[e_7, e_2] &= [e_3, e_8] = [e_5, e_6] = [e_1, e_5] = e_4, \\
[e_8, e_2] &= [e_7, e_3] = [e_6, e_4] = [e_4, e_1] = e_5, \\
[e_2, e_4] &= [e_3, e_5] = [e_8, e_1] = \frac{1}{2}[e_8, e_6] = e_7, \\
[e_2, e_5] &= [e_4, e_3] = [e_1, e_7] = \frac{1}{2}[e_6, e_7] = e_8.
\end{aligned}$$

The normalized Cartan-Killing form $k : \mathfrak{su}_3(\mathbb{C}, 1) \times \mathfrak{su}_3(\mathbb{C}, 1) \rightarrow \mathbb{R}$ is the map $(X, Y) \mapsto \frac{1}{12}\text{trace}(\text{ad}X\text{ad}Y) = \frac{1}{2}\text{trace}(XY)$. An element $X = \lambda_i e_i \in \mathfrak{su}_3(\mathbb{C}, 1)$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, 8$, is elliptic, parabolic or loxodromic according whether

$$k(X) = k(X, X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_6^2 + \lambda_4^2 + \lambda_5^2 + \lambda_7^2 + \lambda_8^2 - 2\lambda_1\lambda_6$$

is smaller, equal or greater 0.

Let H be a connected closed subgroup of the group $PSU_3(\mathbb{C}, 1)$. Then according to [1], Satz 1, p. 251 and [2], Section 5, p. 276, the group H is, up to conjugacy, one of the following:

- (1) H is a subgroup of $Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$,
- (2) H is a subgroup of the 5-dimensional solvable group $NG_{1,1}$ in [1], p. 253,
- (3) H is the group $SL_2(\mathbb{R}) \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$,
- (4) H is the group $SL_2(\mathbb{R}) \times \{1\}/\langle(-1, 1)\rangle \cong PSL_2(\mathbb{R})$,
- (5) H is the connected component of the group $SO_3(\mathbb{R}, 1) \cong PSL_2(\mathbb{R})$.

The Lie algebras \mathfrak{h}_i , $i = 1, \dots, 5$, of H in the cases (1) till (5) are given in this order by

$$\begin{aligned}
\widehat{\mathfrak{h}}_1 &= \langle e_1, e_2, e_3, e_6 \rangle, \quad \widehat{\mathfrak{h}}_2 = \langle e_1 - \frac{1}{2}e_6, e_8, e_4 - e_3, e_5 + e_2, e_6 + e_7 \rangle, \\
\widehat{\mathfrak{h}}_3 &= \langle e_1, e_6, e_7, e_8 \rangle, \quad \widehat{\mathfrak{h}}_4 = \langle e_6, e_7, e_8 \rangle, \quad \widehat{\mathfrak{h}}_5 = \langle e_2, e_4, e_7 \rangle.
\end{aligned}$$

After a straightforward calculation in $\widehat{\mathfrak{h}}_2$ we obtain that the conjugacy classes of the 4-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ are the following:

$$\begin{aligned}
\mathfrak{h}_1 &= \langle e_1, e_2, e_3, e_6 \rangle, \quad \mathfrak{h}_2 = \langle e_4 - e_3, e_2 + e_5, e_6 + e_7, e_8 \rangle, \\
\mathfrak{h}_3 &= \langle e_1 - \frac{1}{2}e_6 + ae_8, e_4 - e_3, e_2 + e_5, e_6 + e_7 \rangle, \quad \mathfrak{h}_4 = \langle e_1, e_6, e_7, e_8 \rangle,
\end{aligned}$$

where $a \in \mathbb{R}$.

Computations in $\widehat{\mathfrak{h}}_1$ and $\widehat{\mathfrak{h}}_2$ yield that the 3-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ have one of the following shapes:

$$\begin{aligned}
\mathfrak{h}_5 &= \langle e_1, e_2, e_3 \rangle, \quad \mathfrak{h}_6 = \langle e_2, e_4, e_7 \rangle, \quad \mathfrak{h}_7 = \langle e_6, e_7, e_8 \rangle, \\
\mathfrak{h}_8 &= \langle e_5 + e_2, e_6 + e_7, e_8 \rangle, \quad \mathfrak{h}_9 = \langle e_4 - e_3 + be_8, e_5 + e_2, e_6 + e_7 \rangle, \\
\mathfrak{h}_{10} &= \langle e_4 - e_3 + b(e_5 + e_2), e_6 + e_7, e_8 + c(e_5 + e_2) \rangle, \\
\mathfrak{h}_{11} &= \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2), e_8 + b(e_4 - e_3) + c(e_5 + e_2), \\
&\quad e_6 + e_7 \rangle, \text{ where } b, c \in \mathbb{R}.
\end{aligned}$$

Similarly we obtain that every 2-dimensional subalgebra of $\mathfrak{su}_3(\mathbb{C}, 1)$ has one of the following forms:

$$\begin{aligned}
\mathbf{h}_{12} &= \langle e_1, e_6 \rangle, \quad \mathbf{h}_{13} = \langle e_4 - e_3, e_6 + e_7 \rangle, \\
\mathbf{h}_{14} &= \langle e_5 + e_2 + b(e_4 - e_3), e_6 + e_7 \rangle, \quad \mathbf{h}_{15} = \langle e_4 - e_3, e_8 + b(e_6 + e_7) \rangle, \\
\mathbf{h}_{16} &= \langle e_5 + e_2 + b(e_4 - e_3), e_8 + c(e_6 + e_7) \rangle, \\
\mathbf{h}_{17} &= \langle e_6 + e_7, e_8 + b(e_4 - e_3) + c(e_5 + e_2) \rangle, \\
\mathbf{h}_{18} &= \langle e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2), e_8 + c(e_4 - e_3) + d(e_5 + e_2) \rangle, \\
\mathbf{h}_{19} &= \langle e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2), e_6 + e_7 \rangle, \\
\mathbf{h}_{20} &= \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7), \\
&\quad e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7) \rangle,
\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ and in the Lie algebra \mathbf{h}_{18} one has $bc - ad = \frac{1}{2}$.

Moreover, every 1-dimensional subalgebra \mathbf{h} of \mathbf{g} is given by

$$\begin{aligned}
\mathbf{h}_{21} &= \langle e_1 + ae_6 \rangle, \quad \mathbf{h}_{22} = \langle e_6 \rangle, \quad \mathbf{h}_{23} = \langle e_8 \rangle, \\
\mathbf{h}_{24} &= \langle e_6 + e_7 + ce_8 \rangle, \quad \mathbf{h}_{25} = \langle e_5 + e_2 + b(e_6 + e_7) + ce_8 \rangle, \\
\mathbf{h}_{26} &= \langle e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle, \\
\mathbf{h}_{27} &= \langle e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle,
\end{aligned}$$

where a, b, c, d are real numbers.

Proposition 9. *The Lie algebra $\mathfrak{su}_3(\mathbb{C}, 1)$ is reductive with a 4-dimensional subalgebra \mathbf{h} and a complementary subspace \mathbf{m} generating \mathbf{g} if and only if the following holds:*

- 1) $\mathbf{h}_1 \cong \mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_2, e_3, e_6 \rangle$ and $\mathbf{m}_1 = \langle e_4, e_5, e_7, e_8 \rangle$,
- 2) $\mathbf{h}_4 \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_6, e_7, e_8 \rangle$ and $\mathbf{m}_4 = \langle e_2, e_3, e_4, e_5 \rangle$.

Proof. For the basis elements of an arbitrary complement \mathbf{m} to \mathbf{h}_1 in \mathbf{g} we have

$$\begin{aligned}
&e_4 + a_1e_1 + b_1e_2 + c_1e_3 + d_1e_6, \quad e_5 + a_2e_1 + b_2e_2 + c_2e_3 + d_2e_6, \\
&e_7 + a_3e_1 + b_3e_2 + c_3e_3 + d_3e_6, \quad e_8 + a_4e_1 + b_4e_2 + c_4e_3 + d_4e_6
\end{aligned}$$

with the real numbers a_i, b_i, c_i, d_i , $i = 1, 2, 3, 4$.

An arbitrary complement \mathbf{m} to \mathbf{h}_2 in \mathbf{g} has as generators

$$\begin{aligned}
&e_1 + a_1(e_4 - e_3) + b_1(e_5 + e_2) + c_1(e_6 + e_7) + d_1e_8, \\
&e_2 + a_2(e_4 - e_3) + b_2(e_5 + e_2) + c_2(e_6 + e_7) + d_2e_8, \\
&e_3 + a_3(e_4 - e_3) + b_3(e_5 + e_2) + c_3(e_6 + e_7) + d_3e_8, \\
&e_6 + a_4(e_4 - e_3) + b_4(e_5 + e_2) + c_4(e_6 + e_7) + d_4e_8,
\end{aligned}$$

where $a_i, b_i, c_i, d_i, i = 1, 2, 3, 4$ are real parameters.

The basis elements of an arbitrary complement \mathbf{m} to \mathbf{h}_3 in \mathbf{g} are

$$\begin{aligned} e_3 + a_1(e_1 - \frac{1}{2}e_6 + ae_8) + b_1(e_4 - e_3) + c_1(e_2 + e_5) + d_1(e_6 + e_7), \\ e_5 + a_2(e_1 - \frac{1}{2}e_6 + ae_8) + b_2(e_4 - e_3) + c_2(e_2 + e_5) + d_2(e_6 + e_7), \\ e_7 + a_3(e_1 - \frac{1}{2}e_6 + ae_8) + b_3(e_4 - e_3) + c_3(e_2 + e_5) + d_3(e_6 + e_7), \\ e_8 + a_4(e_1 - \frac{1}{2}e_6 + ae_8) + b_4(e_4 - e_3) + c_4(e_2 + e_5) + d_4(e_6 + e_7), \end{aligned}$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1, 2, 3, 4$.

As the generators of an arbitrary complement \mathbf{m} to \mathbf{h}_4 in \mathbf{g} we can choose the following:

$$\begin{aligned} e_2 + a_1e_1 + b_1e_6 + c_1e_7 + d_1e_8, \quad e_3 + a_2e_1 + b_2e_6 + c_2e_7 + d_2e_8, \\ e_4 + a_3e_1 + b_3e_6 + c_3e_7 + d_3e_8, \quad e_5 + a_4e_1 + b_4e_6 + c_4e_7 + d_4e_8, \end{aligned}$$

where $a_i, b_i, c_i, d_i, i = 1, 2, 3, 4$ are real numbers.

Now the assertion follows from the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. \square

Proposition 10. *The Lie algebra $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ is reductive with respect to precisely one of the following pairs (\mathbf{h}, \mathbf{m}) , where \mathbf{h} is a 3-dimensional subalgebra of \mathbf{g} and \mathbf{m} is a complementary subspace to \mathbf{h} generating \mathbf{g} :*

- 1) $\mathbf{h}_6 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_2, e_4, e_7 \rangle$ and $\mathbf{m}_6 = \langle e_1, e_3, e_5, e_6, e_8 \rangle$,
- 2) $\mathbf{h}_7 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_6, e_7, e_8 \rangle$ and $\mathbf{m}_7 = \langle e_1 - \frac{1}{2}e_6, e_2, e_3, e_4, e_5 \rangle$.

Proof. An arbitrary complement \mathbf{m}_i to the subalgebra $\mathbf{h}_i, i = 5, \dots, 11$, in \mathbf{g} has as generators in the case $i = 5$

$$\begin{aligned} e_4 + a_1e_1 + b_1e_2 + c_1e_3, \quad e_5 + a_2e_1 + b_2e_2 + c_2e_3, \quad e_6 + a_3e_1 + b_3e_2 + c_3e_3, \\ e_7 + a_4e_1 + b_4e_2 + c_4e_3, \quad e_8 + a_5e_1 + b_5e_2 + c_5e_3, \end{aligned}$$

in the case $i = 6$

$$\begin{aligned} e_1 + a_1e_2 + b_1e_4 + c_1e_7, \quad e_3 + a_2e_2 + b_2e_4 + c_2e_7, \quad e_5 + a_3e_2 + b_3e_4 + c_3e_7, \\ e_6 + a_4e_2 + b_4e_4 + c_4e_7, \quad e_8 + a_5e_2 + b_5e_4 + c_5e_7, \end{aligned}$$

in the case $i = 7$

$$\begin{aligned} e_1 + a_1e_6 + b_1e_7 + c_1e_8, \quad e_2 + a_2e_6 + b_2e_7 + c_2e_8, \quad e_3 + a_3e_6 + b_3e_7 + c_3e_8, \\ e_4 + a_4e_6 + b_4e_7 + c_4e_8, \quad e_5 + a_5e_6 + b_5e_7 + c_5e_8, \end{aligned}$$

in the case $i = 8$

$$\begin{aligned} e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1e_8, \quad e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2e_8, \\ e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3e_8, \quad e_4 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4e_8, \\ e_6 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5e_8, \end{aligned}$$

in the case $i = 9$

$$\begin{aligned}
& e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1(e_4 - e_3 + be_8), \\
& e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2(e_4 - e_3 + be_8), \\
& e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3(e_4 - e_3 + be_8), \\
& e_6 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4(e_4 - e_3 + be_8), \\
& e_8 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5(e_4 - e_3 + be_8),
\end{aligned}$$

in the case $i = 10$

$$\begin{aligned}
& e_1 + a_1(e_4 - e_3 + b(e_2 + e_5)) + b_1(e_6 + e_7) + c_1(e_8 + c(e_2 + e_5)), \\
& e_2 + a_2(e_4 - e_3 + b(e_2 + e_5)) + b_2(e_6 + e_7) + c_2(e_8 + c(e_2 + e_5)), \\
& e_3 + a_3(e_4 - e_3 + b(e_2 + e_5)) + b_3(e_6 + e_7) + c_3(e_8 + c(e_2 + e_5)), \\
& e_5 + a_4(e_4 - e_3 + b(e_2 + e_5)) + b_4(e_6 + e_7) + c_4(e_8 + c(e_2 + e_5)), \\
& e_6 + a_5(e_4 - e_3 + b(e_2 + e_5)) + b_5(e_6 + e_7) + c_5(e_8 + c(e_2 + e_5)),
\end{aligned}$$

and in the case $i = 11$

$$\begin{aligned}
& e_2 + a_1(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + \\
& b_1(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_1(e_6 + e_7), \\
& e_3 + a_2(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + \\
& b_2(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_2(e_6 + e_7), \\
& e_4 + a_3(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + \\
& b_3(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_3(e_6 + e_7), \\
& e_5 + a_4(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + \\
& b_4(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_4(e_6 + e_7), \\
& e_7 + a_5(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + \\
& b_5(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_5(e_6 + e_7),
\end{aligned}$$

where $a_j, b_j, c_j \in \mathbb{R}$, $j = 1, \dots, 5$. The relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 5, \dots, 11$, yields the assertion. \square

Proposition 11. *The Lie algebra $\mathfrak{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ is reductive with respect to the following pairs (\mathbf{h}, \mathbf{m}) , where \mathbf{h} is a 2-dimensional subalgebra of \mathfrak{g} and \mathbf{m} is a complementary subspace to \mathbf{h} generating \mathfrak{g} , if and only if one of the following holds:*

- 1) $\mathbf{h}_{12} = \langle e_1, e_6 \rangle$ and $\mathbf{m}_{12} = \langle e_2, e_3, e_4, e_5, e_7, e_8 \rangle$,
- 2) $\mathbf{h}_{20} = \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7),$
 $e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7) \rangle$

and

$$\mathbf{m}_{20} = \langle e_6 + e_7, e_2 + e_5, e_4 - e_3, e_4 - be_8 + 2ae_1 - ae_6, e_2 + ae_8 + 2be_1 - be_6, \\
e_6 + ce_8 + be_5 - ae_4 \rangle, \quad a, b, c \in \mathbb{R}.$$

Proof. An arbitrary complement \mathbf{m}_i to the subalgebra \mathbf{h}_i , $i = 12, \dots, 20$, in \mathbf{g} has as generators in the case $i = 12$

$$\begin{aligned} e_2 + a_1 e_1 + b_1 e_6, & \quad e_3 + a_2 e_1 + b_2 e_6, & \quad e_4 + a_3 e_1 + b_3 e_6, \\ e_5 + a_4 e_1 + b_4 e_6, & \quad e_7 + a_5 e_1 + b_5 e_6, & \quad e_8 + a_6 e_1 + b_6 e_6, \end{aligned}$$

in the case $i = 13$

$$\begin{aligned} e_1 + a_1(e_4 - e_3) + b_1(e_6 + e_7), & \quad e_2 + a_2(e_4 - e_3) + b_2(e_6 + e_7), \\ e_3 + a_3(e_4 - e_3) + b_3(e_6 + e_7), & \quad e_5 + a_4(e_4 - e_3) + b_4(e_6 + e_7), \\ e_6 + a_5(e_4 - e_3) + b_5(e_6 + e_7), & \quad e_8 + a_6(e_4 - e_3) + b_6(e_6 + e_7), \end{aligned}$$

in the case $i = 14$

$$\begin{aligned} e_1 + a_1(e_2 + e_5 + b(e_4 - e_3)) + b_1(e_6 + e_7), \\ e_2 + a_2(e_2 + e_5 + b(e_4 - e_3)) + b_2(e_6 + e_7), \\ e_3 + a_3(e_2 + e_5 + b(e_4 - e_3)) + b_3(e_6 + e_7), \\ e_4 + a_4(e_2 + e_5 + b(e_4 - e_3)) + b_4(e_6 + e_7), \\ e_6 + a_5(e_2 + e_5 + b(e_4 - e_3)) + b_5(e_6 + e_7), \\ e_8 + a_6(e_2 + e_5 + b(e_4 - e_3)) + b_6(e_6 + e_7), \end{aligned}$$

in the case $i = 15$

$$\begin{aligned} e_1 + a_1(e_4 - e_3) + b_1(e_8 + b(e_6 + e_7)), \\ e_2 + a_2(e_4 - e_3) + b_2(e_8 + b(e_6 + e_7)), \\ e_3 + a_3(e_4 - e_3) + b_3(e_8 + b(e_6 + e_7)), \\ e_5 + a_4(e_4 - e_3) + b_4(e_8 + b(e_6 + e_7)), \\ e_6 + a_5(e_4 - e_3) + b_5(e_8 + b(e_6 + e_7)), \\ e_7 + a_6(e_4 - e_3) + b_6(e_8 + b(e_6 + e_7)), \end{aligned}$$

in the case $i = 16$

$$\begin{aligned} e_1 + a_1(e_5 + e_2 + b(e_4 - e_3)) + b_1(e_8 + c(e_6 + e_7)), \\ e_2 + a_2(e_5 + e_2 + b(e_4 - e_3)) + b_2(e_8 + c(e_6 + e_7)), \\ e_3 + a_3(e_5 + e_2 + b(e_4 - e_3)) + b_3(e_8 + c(e_6 + e_7)), \\ e_4 + a_4(e_5 + e_2 + b(e_4 - e_3)) + b_4(e_8 + c(e_6 + e_7)), \\ e_6 + a_5(e_5 + e_2 + b(e_4 - e_3)) + b_5(e_8 + b(e_6 + e_7)), \\ e_7 + a_6(e_5 + e_2 + b(e_4 - e_3)) + b_6(e_8 + b(e_6 + e_7)), \end{aligned}$$

in the case $i = 17$

$$e_1 + a_1(e_6 + e_7) + b_1(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$

$$\begin{aligned}
& e_2 + a_2(e_6 + e_7) + b_2(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \\
& e_3 + a_3(e_6 + e_7) + b_3(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \\
& e_4 + a_4(e_6 + e_7) + b_4(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \\
& e_5 + a_5(e_6 + e_7) + b_5(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \\
& e_6 + a_6(e_6 + e_7) + b_6(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),
\end{aligned}$$

in the case $i = 18$

$$\begin{aligned}
& e_1 + a_1(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_1(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \\
& e_2 + a_2(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_2(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \\
& e_3 + a_3(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_3(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \\
& e_4 + a_4(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_4(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \\
& e_5 + a_5(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_5(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \\
& e_6 + a_6(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_6(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),
\end{aligned}$$

in the case $i = 19$

$$\begin{aligned}
& e_2 + a_1(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_1(e_6 + e_7), \\
& e_3 + a_2(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_2(e_6 + e_7), \\
& e_4 + a_3(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_3(e_6 + e_7), \\
& e_5 + a_4(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_4(e_6 + e_7), \\
& e_7 + a_5(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_5(e_6 + e_7), \\
& e_8 + a_6(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_6(e_6 + e_7),
\end{aligned}$$

and in the case $i = 20$

$$\begin{aligned}
& e_2 + a_1(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7)) + \\
& + b_1(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \\
& e_3 + a_2(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7)) + \\
& + b_2(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \\
& e_4 + a_3(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7)) + \\
& + b_3(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \\
& e_5 + a_4(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7)) + \\
& + b_4(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \\
& e_6 + a_5(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7)) + \\
& + b_5(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \\
& e_7 + a_6(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7)) + \\
& + b_6(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),
\end{aligned}$$

where $a_j, b_j \in \mathbb{R}$, $j = 1, \dots, 6$. Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 12, \dots, 20$, we obtain the assertion. \square

Proposition 12. *The Lie algebra $\mathfrak{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ is reductive with a 1-dimensional subalgebra \mathfrak{h} and a 7-dimensional complementary subspace \mathfrak{m} generating \mathfrak{g} in precisely one of the following cases:*

- 1) $\mathfrak{h} = \langle e_1 - 2e_6 \rangle$, $\mathfrak{m}_{b,c,d} = \langle e_2 + b(e_1 - 2e_6), e_3 + c(e_1 - 2e_6), e_6 + d(e_1 - 2e_6), e_4, e_5, e_7, e_8 \rangle$, when $b, c, d \in \mathbb{R}$,
- 2) $\mathfrak{h} = \langle e_1 + e_6 \rangle$ and $\mathfrak{m}_{b,c,d} = \langle e_2, e_3, e_7, e_8, e_4 + d(e_1 + e_6), e_5 + b(e_1 + e_6), e_6 + c(e_1 + e_6) \rangle$ with $b, c, d \in \mathbb{R}$,
- 3) $\mathfrak{h} = \langle e_1 - \frac{1}{2}e_6 \rangle$, $\mathfrak{m}_{b,c,d} = \langle e_2, e_3, e_4, e_5, e_6 + b(e_1 - \frac{1}{2}e_6), e_7 + c(e_1 - \frac{1}{2}e_6), e_8 + d(e_1 - \frac{1}{2}e_6) \rangle$ and $b, c, d \in \mathbb{R}$,
- 4) $\mathfrak{h}_a = \langle e_1 + ae_6 \rangle$ and $\mathfrak{m}_b = \langle e_2, e_3, e_4, e_5, e_6 + b(e_1 + ae_6), e_7, e_8 \rangle$, where $a \in \mathbb{R} \setminus \{-\frac{1}{2}, -2, 1\}$, $b, c, d \in \mathbb{R}$,
- 5) $\mathfrak{h} = \langle e_6 \rangle$ and $\mathfrak{m}_a = \langle e_1 + ae_6, e_2, e_3, e_4, e_5, e_7, e_8 \rangle$, $a \in \mathbb{R}$,
- 6) $\mathfrak{h} = \langle e_8 \rangle$ and $\mathfrak{m}_a = \langle e_1 + ae_8, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$, $a \in \mathbb{R}$,
- 7) $\mathfrak{h} = \langle e_6 + e_7 + ce_8 \rangle$ and $\mathfrak{m}_b = \langle e_1 + bce_8, e_2, e_3, e_4, e_5, e_6 + e_7, e_7 - \frac{1}{c}e_8 \rangle$ with $c \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$,
- 8) $\mathfrak{h}_{b,c} = \langle e_5 + e_2 + b(e_6 + e_7) + ce_8 \rangle$ and $\mathfrak{m}_d = \langle e_1 - \frac{c^3d - cd - b}{2c}e_8, e_2 + \frac{1}{c}e_8, e_3 + cde_8, e_7 - \frac{b + cd}{c}e_8, e_4 - e_3, e_2 + e_5, e_6 + e_7 \rangle$, where $b, c, d \in \mathbb{R}, c \neq 0$,
- 9) $\mathfrak{h}_{a,b,c} = \langle e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle$ and $\mathfrak{m}_d = \langle e_2 - dce_8, e_3 - \frac{1 + dc^2a + a^2}{c}e_8, e_6 - \frac{a^3 + a - bc + dc^2 + dc^2a^2}{c^2}e_8, e_5 + e_2, e_6 + e_7, e_4 - e_3, e_1 + \frac{bc + c^2a - a - a^3 + c^4d - c^2d - c^2a^2d}{2c^2}e_8 \rangle$ with $a, b, c, d \in \mathbb{R}, c \neq 0$,
- 10) $\mathfrak{h}_{a,b,c,d} = \langle e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle$ and $\mathfrak{m}_f = \langle e_6 + e_7, e_4 - e_3, e_5 + e_2, e_3 + f(e_1 - \frac{1}{2}e_6 + ce_8), e_2 - \frac{2c}{3}e_4 - \frac{4a}{3}e_1 - \frac{2a}{3}e_7 + \frac{2d}{3}e_8, e_7 - \frac{b}{c}e_8 + \frac{a}{c}e_4 + \frac{d}{c}e_2, e_8 - \frac{8ac - 4fc^2 + 24fd^2 - 9f + 12d}{2(8dc - 3a + 4ac^2)}(e_1 - \frac{1}{2}e_6 + ce_8) \rangle$, where $a, b, c, d, f \in \mathbb{R}, c \neq 0, 8dc - 3a + 4ac^2 \neq 0$.

Proof. An arbitrary complement \mathfrak{m}_i to the subalgebra \mathfrak{h}_i , $i = 21, \dots, 27$, in \mathfrak{g} has as generators in the case $i = 21$

$$e_2 + a_1(e_1 + ae_6), \quad e_3 + a_2(e_1 + ae_6), \quad e_4 + a_3(e_1 + ae_6), \quad e_5 + a_4(e_1 + ae_6), \\ e_6 + a_5(e_1 + ae_6), \quad e_7 + a_6(e_1 + ae_6), \quad e_8 + a_7(e_1 + ae_6),$$

in the case $i = 22$

$$e_1 + a_1e_6, \quad e_2 + a_2e_6, \quad e_3 + a_3e_6, \quad e_4 + a_4e_6, \\ e_5 + a_5e_6, \quad e_7 + a_6e_6, \quad e_8 + a_7e_6,$$

in the case $i = 23$

$$e_1 + a_1e_8, \quad e_2 + a_2e_8, \quad e_3 + a_3e_8, \quad e_4 + a_4e_8, \\ e_5 + a_5e_8, \quad e_6 + a_6e_8, \quad e_7 + a_7e_8,$$

in the case $i = 24$

$$\begin{aligned} e_1 + a_1(e_6 + e_7 + ce_8), \quad e_2 + a_2(e_6 + e_7 + ce_8), \\ e_3 + a_3(e_6 + e_7 + ce_8), \quad e_4 + a_4(e_6 + e_7 + ce_8), \\ e_5 + a_5(e_6 + e_7 + ce_8), \quad e_7 + a_6(e_6 + e_7 + ce_8), \\ e_8 + a_7(e_6 + e_7 + ce_8), \end{aligned}$$

in the case $i = 25$

$$\begin{aligned} e_1 + a_1(e_5 + e_2 + b(e_6 + e_7) + ce_8), \quad e_2 + a_2(e_5 + e_2 + b(e_6 + e_7) + ce_8), \\ e_3 + a_3(e_5 + e_2 + b(e_6 + e_7) + ce_8), \quad e_4 + a_4(e_5 + e_2 + b(e_6 + e_7) + ce_8), \\ e_6 + a_5(e_5 + e_2 + b(e_6 + e_7) + ce_8), \quad e_7 + a_6(e_5 + e_2 + b(e_6 + e_7) + ce_8), \\ e_8 + a_7(e_5 + e_2 + b(e_6 + e_7) + ce_8), \end{aligned}$$

in the case $i = 26$

$$\begin{aligned} e_1 + a_1(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_2 + a_2(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_3 + a_3(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_5 + a_4(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_6 + a_5(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_7 + a_6(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_8 + a_6(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \end{aligned}$$

in the case $i = 27$

$$\begin{aligned} e_2 + a_1(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_3 + a_2(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_4 + a_3(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_5 + a_4(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_6 + a_5(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_7 + a_6(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \\ e_8 + a_7(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8), \end{aligned}$$

where a_j , $j = 1, \dots, 7$, are real parameters. The relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 21, \dots, 27$, yields the assertion. \square

4 Left A-loops as sections in simple Lie groups

The connected almost differentiable left A-loops L with $\dim L \leq 2$ are classified in [18], Section 27 and Theorem 18.14. Furthermore, all 3-dimensional left A-loops which are differentiable sections in a non-solvable Lie group are determined in [6]. In this section we deal with the at least 4-dimensional almost differentiable left A-loops having an at most 9-dimensional simple Lie group G as the group topologically generated by their left translations. According to Lemma 1 the group G is not compact.

Proposition 13. *There exists no at least 4-dimensional differentiable left A-loop having a group locally isomorphic to $PSL_2(\mathbb{C})$ as the group topologically generated by its left translations.*

Proof. Since the tangent space $T_e L$ for an almost differentiable left A-loop L is reductive only the pairs (\mathbf{h}, \mathbf{m}) in Proposition 4 can occur as the tangent objects $(T_1 H, T_e L)$, where H is the stabilizer of the identity e of L . A maximal compact subalgebra of the Lie algebra \mathbf{h}_3 as well as of \mathbf{h}_6 is isomorphic to $so_2(\mathbb{R})$. Hence the Lie group corresponding to \mathbf{h}_3 as well as to \mathbf{h}_6 cannot be the stabilizer of $e \in L$ (cf. Lemma 2). Moreover, the hyperbolic elements $e_1 \in \mathbf{h}_4$ and $e_2 \in \mathbf{m}_a$ are conjugate (see 1.1). This contradiction to Lemma 3 yields the assertion. \square

Proposition 14. *Let G be locally isomorphic to $SL_3(\mathbb{R})$. Every connected almost differentiable left A-loop having G as the group topologically generated by its left translations is isomorphic to the 5-dimensional Bruck loop L_0 of hyperbolic type having the group $SO_3(\mathbb{R})$ as the stabilizer of $e \in L_0$.*

Proof. Since the tangent space $T_e L$ for an almost differentiable left A-loop L is reductive we have to investigate the pairs (\mathbf{h}, \mathbf{m}) listed in Propositions 5, 6, 7 and 8. According to Lemma 2 the Lie groups belonging to the Lie algebras \mathbf{h}_5 , \mathbf{h}_7 , \mathbf{h}_8 , \mathbf{h}_{30} and \mathbf{h}_{35} for $b = 0$ cannot be stabilizers of $e \in L$. The element

$-e_5 + e_8 \in \mathbf{h}_{26}$ is conjugate to $\frac{1}{2}e_1 + 2e_3 \in \mathbf{m}_{26}$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$,

the element $e_2 + e_8 \in \mathbf{h}_{32}$ is conjugate to $e_1 + 2e_7 - e_8 + 2e_4 \in \mathbf{m}_d$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 2 & 0 \end{pmatrix}$ and $e_6 - e_7 + b(e_5 + e_8) \in \mathbf{h}_{35}$, $b > 0$, is conjugate to

$(b^2 + 1)e_1 - e_3 + 2b(e_5 - e_8) \in \mathbf{m}_c$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -b & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Moreover, the

element $e_8 + \frac{1}{a}e_5 \in \mathbf{h}_{31,1}$ is conjugate to $\frac{-a^2+a+1}{a^2}e_1 + e_2 + e_3 + e_4 - e_6 - e_7 \in \mathbf{m}_b$ under $g = \begin{pmatrix} 1 & -\frac{1}{a} & -1 \\ 1 & \frac{a+1}{a} & -1 \\ 0 & \frac{a}{2+a} & \frac{a}{2+a} \end{pmatrix}$.

In the case 2) of Proposition 8 we choose $k \in \mathbb{R} \setminus \{0\}$ in such a way that $l := k^2c + k + b \neq 0$. Then the element $l(e_5 - 2e_8) \in \mathbf{h}_{31,2}$ is conjugate to $e_1 + b(e_5 - 2e_8) + 3l(e_2 - ke_6) + k(e_3 + c(e_5 - 2e_8)) + \frac{3k^2c+k+3b}{3l}(ke_4 - e_7) \in \mathbf{m}_{b,c,d}$ under $g = \begin{pmatrix} 0 & -\frac{3k^2c+k+3b}{3kl} & 1 \\ k & 1 & 0 \\ -\frac{k}{3l} & \frac{1}{3l} & 1 \end{pmatrix}$.

In the case 3) of Proposition 8 we take $k \in \mathbb{R}$ such that $n := k^2b - 2k + c \neq 0$. Then the element $n(e_5 - \frac{1}{2}e_8) \in \mathbf{h}_{31,3}$ is conjugate to $-ke_1 + k^2(e_2 + b(e_5 - \frac{1}{2}e_8)) + \frac{3k^2b-2k+3c}{2}(e_3 - ke_6) + e_4 + c(e_5 - \frac{1}{2}e_8) + e_7 \in \mathbf{m}_{b,c,d}$ under $g = \begin{pmatrix} 0 & \frac{2}{3n} & \frac{-3k^2b+2b-3c}{3n} \\ 1 & 1 & -k \\ 1 & 0 & k \end{pmatrix}$.

In the case 4) of Proposition 8 we take $k \in \mathbb{R}$ such that $m := k^2b + k + c \neq 0$. Then the element $m(e_5 + e_8) \in \mathbf{h}_{31,4}$ is conjugate to $(3c + 3k^2b + k)(ke_2 - e_1) + e_4 - ke_3 + e_7 + c(e_5 + e_8) + k^2(e_6 + b(e_5 + e_8)) \in \mathbf{m}_{b,c,d}$ under $g = \begin{pmatrix} 1 & 1 & -k \\ -\frac{1}{3m} & 0 & \frac{-3c-k-3k^2b}{3m} \\ 0 & 1 & k \end{pmatrix}$. These facts contradict Lemma

3.

In the remaining case one has $[\mathbf{m}_6, \mathbf{m}_6] = \mathbf{h}_6$ and the loop L with $T_e L = \mathbf{m}_6$ is a Bruck loop. The assertion follows now from the proof of the Theorem 13 in [5], p. 12. \square

Since the exponential image of the Lie algebra $\mathfrak{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ is much more complicated than the exponential image of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ we treat the almost differentiable left A-loops having $PSU_3(\mathbb{C}, 1)$ as the group topologically generated by the left translations under the assumption that their dimension is at most 5.

Proposition 15. *Let G be locally isomorphic to $PSU_3(\mathbb{C}, 1)$. Every at most 5-dimensional connected almost differentiable left A-loop having G as the group topologically generated by the left translations is isomorphic to the complex hyperbolic plane loop L_0 having the group $Spin_3 \times SO_2(\mathbb{R}) / \langle (-1, -1) \rangle$ as the stabilizer of $e \in L_0$.*

Proof. Since the tangent space $T_e L$ for an almost differentiable left A-loop L is reductive we have to deal only with the pairs (\mathbf{h}, \mathbf{m}) described in the Propositions 9, 10. The complex hyperbolic plane loop L_0 is realized on the exponential image of the subspace \mathbf{m}_1 (cf. [5], p. 8). The Lie group corresponding to \mathbf{h}_4 cannot be the stabilizer of a 4-dimensional topological loop L (see Lemma 2). According to 1.2 the element $e_2 \in \mathbf{h}_6$ is conjugate to $e_1 \in \mathbf{m}_6$, which is a contradiction to Lemma 3. Two loxodromic elements of $\mathfrak{su}_3(\mathbb{C}, 1)$ are conjugate in $SU_3(\mathbb{C}, 1)$ if and only if they have the same

eigenvalues (cf. Prop. 3.2.3 (d) in [3], p. 65) and therefore they are conjugate in $SL_3(\mathbb{C})$. Since the elements $e_7 \in \mathfrak{h}_7$ and $e_4 \in \mathfrak{m}_7$ are loxodromic and $Ad_g(e_7) = e_4$ with $g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 1 \end{pmatrix} \in SL_3(\mathbb{C})$ we have also a contradiction to Lemma 3. \square

At the end of this section we show that several reductive spaces $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m})$, where $\mathfrak{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ and $\dim \mathfrak{h} \leq 2$ can not correspond to an almost differentiable left A-loop.

Proposition 16. *There is no almost differentiable left A-loop corresponding to one of the following triples: $(\mathfrak{g}, \mathfrak{h}_{12}, \mathfrak{m}_{12})$ in Proposition 11 and $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}_a)$ in the case 6) as well as $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}_b)$ in the case 7) of Proposition 12.*

Proof. Since the elements $e_1 \in \mathfrak{h}_{12}$ and $e_2 \in \mathfrak{m}_{12}$ are elliptic in a subalgebra isomorphic to $\mathfrak{so}_3(\mathbb{R})$ of \mathfrak{g} (see 1.2) they are conjugate under $Ad PSU_3(\mathbb{C}, 1)$. Since the element $e_8 \in \mathfrak{h}$ in the case 6 as well as $e_6 + e_7 + ce_8 \in \mathfrak{h}$, $c \neq 0$, in the case 7 of Proposition 12 is hyperbolic in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ of \mathfrak{g} (see 1.1), we have that e_8 and $e_7 \in \mathfrak{m}_a$ respectively $e_6 + e_7 + ce_8$ and $-\frac{1}{\sqrt{2+2c^2}}(e_7 - \frac{1}{c}e_8) \in \mathfrak{m}_b$ are conjugate under $Ad PSU_3(\mathbb{C}, 1)$. This contradicts Lemma 3. \square

5 Reductive loops corresponding to semi-simple Lie groups of dimension 6

Let $G = G_1 \times G_2$ be the group topologically generated by the left translations of a connected almost differentiable left A-loop L , such that G_i , $i = 1, 2$, is a 3-dimensional quasi-simple Lie group. In contrast to the non-existence of 3-dimensional almost differentiable left A-loops belonging to G (cf. Propositions 5 and 8 in [6]) we will show that there are such loops L with $G = G_1 \times G_2$ as the group topologically generated by the left translations if $\dim L \geq 4$.

The following fact is well known from linear algebra:

Lemma 17. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where \mathfrak{g}_i , $i = 1, 2$ are simple Lie algebras of dimension 3. For any subspace \mathfrak{m} with dimension 4 respectively 5 the intersections $\mathfrak{m} \cap \mathfrak{g}_1$ and $\mathfrak{m} \cap \mathfrak{g}_2$ have dimension at least 1 respectively at least 2.*

The fact that the coset space G/H is parallelizable is reflected in the following lemma.

Lemma 18. *Let G be isomorphic to the Lie group $G_1 \times G_2$, such that $G_2 \cong SO_3(\mathbb{R})$ and for the subgroup H of G one has $H = H_1 \times H_2$ with $1 \neq H_2 \leq G_2$. Then G cannot be the group topologically generated by the left translations of a topological loop.*

For the proof see Lemma 2 in [5], p. 5.

First let G be locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. Since the at most 2-dimensional connected subgroups of G are tori and $\dim L \geq 4$ Lemma 2 gives

Proposition 19. *There is no left A -loop as differentiable section in a group locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$.*

Now let G be locally isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where G_2 is either the group $SO_3(\mathbb{R})$ or $PSL_2(\mathbb{R})$. Using the real basis of $\mathfrak{sl}_2(\mathbb{R})$ respectively of $\mathfrak{so}_3(\mathbb{R})$ introduced in 1.1 respectively in 1.2 we can choose $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$, $(0, \varepsilon e_1)$, $(0, \varepsilon e_2)$, $(0, \varepsilon e_3)$ as a real basis of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2$, where $\varepsilon = i$ with $i^2 = -1$ for $\mathfrak{g}_2 = \mathfrak{so}_3(\mathbb{R})$ and $\varepsilon = 1$ for $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.

Denote by H a subgroup of G . First we assume that H is decomposable into a direct product. If H has dimension 2 then with Lemma 18 we obtain that H is (up to interchanging the components) either $\mathcal{L}_2 \times \{1\}$ or $K_1 \times K_2$, where K_i , $i = 1, 2$ are 1-dimensional subgroups of $PSL_2(\mathbb{R})$. Now according to 1.1 the Lie algebra \mathfrak{h} of H has one of the following forms:

$$\mathfrak{h}_1 = \langle (e_3, 0), (0, e_3) \rangle, \quad \mathfrak{h}_2 = \langle (e_3, 0), (0, e_2 + e_3) \rangle, \quad \mathfrak{h}_3 = \langle (e_3, 0), (0, e_1) \rangle,$$

$$\mathfrak{h}_4 = \langle (e_1, 0), (0, e_1) \rangle, \quad \mathfrak{h}_5 = \langle (e_1, 0), (0, e_2 + e_3) \rangle,$$

$$\mathfrak{h}_6 = \langle (e_2 + e_3, 0), (0, e_2 + e_3) \rangle, \quad \mathfrak{h}_7 = \langle (e_1, 0), (e_2 + e_3, 0) \rangle.$$

The Lie algebras \mathfrak{h}_1 till \mathfrak{h}_7 are subalgebras of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ but \mathfrak{h}_7 is also a subalgebra of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.

If $\dim H = 1$ then H has the shape $K_1 \times \{1\}$ with a 1-dimensional subgroup K_1 of $PSL_2(\mathbb{R})$. Then according to 1.1 the Lie algebra \mathfrak{h} of H has (up to interchanging the components) one of the following forms:

$$\mathfrak{h}_8 = \langle (e_3, 0) \rangle, \quad \mathfrak{h}_9 = \langle (e_1, 0) \rangle, \quad \mathfrak{h}_{10} = \langle (e_2 + e_3, 0) \rangle.$$

These algebras are subalgebras of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ as well as $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.

Now we suppose that H is not a direct product of two subgroups. In the case $\dim H = 2$ one has $H = \{(x, \varphi(x)) \mid x \in \mathcal{L}_2\}$, where $\varphi \neq 1$ is a homomorphism of \mathcal{L}_2 into $PSL_2(\mathbb{R})$. If φ is injective then the Lie algebra of H is a subalgebra of $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ and has the shape

$$\mathfrak{h}_{11} = \langle (e_1, e_1), (e_2 + e_3, e_2 + e_3) \rangle.$$

If φ has 1-dimensional kernel then the Lie algebra of H is given by

$$\mathfrak{h}_{12} = \langle (e_1, k), (e_2 + e_3, 0) \rangle,$$

where k denotes either the element e_1 or $e_2 + e_3$ of $\mathfrak{sl}_2(\mathbb{R})$ or e_3 of $\mathfrak{sl}_2(\mathbb{R}) \cap \mathfrak{so}_3(\mathbb{R})$ (see 1.1 and 1.2).

In the case $\dim H = 1$ one has $H = \{(k_1, \varphi(k_1)) \mid k_1 \in K_1\}$, where K_1 is a 1-dimensional subgroup of $PSL_2(\mathbb{R})$ and $\varphi \neq 1$ is a homomorphism of K_1 into $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. Then the Lie algebra \mathfrak{h} of H has (up to interchanging the components) one of the following forms:

$$\begin{aligned} \mathfrak{h}_{13} &= \langle (e_1, e_1) \rangle, & \mathfrak{h}_{14} &= \langle (e_1, e_2 + e_3) \rangle, & \mathfrak{h}_{15} &= \langle (e_2 + e_3, e_2 + e_3) \rangle, \\ \mathfrak{h}_{16} &= \langle (e_1, e_3) \rangle, & \mathfrak{h}_{17} &= \langle (e_2 + e_3, e_3) \rangle, & \mathfrak{h}_{18} &= \langle (e_3, e_3) \rangle. \end{aligned}$$

The Lie algebra \mathfrak{h}_{13} till \mathfrak{h}_{18} are subalgebras of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ but $\mathfrak{h}_{16}, \mathfrak{h}_{17}, \mathfrak{h}_{18}$ are also subalgebras of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.

Proposition 20. *The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2$, where \mathfrak{g}_2 is a 3-dimensional simple Lie algebra, is reductive with an at most 2-dimensional subalgebra \mathfrak{h} and a complementary subspace \mathfrak{m} generating \mathfrak{g} in exactly one of the following cases:*

- 1) $\mathfrak{h}_8 = \langle (e_3, 0) \rangle$, $\mathfrak{m}_a = \langle (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (ae_3, e_3) \rangle$,
- 2) $\mathfrak{h}_8 = \langle (e_3, 0) \rangle$, $\mathfrak{m}_b = \langle (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (be_3, \varepsilon e_2), (0, e_3) \rangle$,
- 3) $\mathfrak{h}_8 = \langle (e_3, 0) \rangle$, $\mathfrak{m}_c = \langle (e_1, 0), (e_2, 0), (ce_3, \varepsilon e_1), (0, \varepsilon e_2), (0, e_3) \rangle$,
- 4) $\mathfrak{h}_9 = \langle (e_1, 0) \rangle$, $\mathfrak{m}_d = \langle (e_2, 0), (e_3, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (de_1, e_3) \rangle$,
- 5) $\mathfrak{h}_9 = \langle (e_1, 0) \rangle$, $\mathfrak{m}_f = \langle (e_2, 0), (e_3, 0), (0, \varepsilon e_1), (fe_1, \varepsilon e_2), (0, e_3) \rangle$,
- 6) $\mathfrak{h}_9 = \langle (e_1, 0) \rangle$, $\mathfrak{m}_g = \langle (e_2, 0), (e_3, 0), (ge_1, \varepsilon e_1), (0, \varepsilon e_2), (0, e_3) \rangle$,
- 7) $\mathfrak{h}_{16} = \langle (e_1, e_3) \rangle$, $\mathfrak{m}_h = \langle (e_2, 0), (e_3, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (he_1, (1+h)e_3) \rangle$,
- 8) $\mathfrak{h}_{17} = \langle (e_2 + e_3, e_3) \rangle$, $\mathfrak{m}_k = \langle (e_3, ke_3), (e_1, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (e_2 + e_3, 0) \rangle$,
- 9) $\mathfrak{h}_{18} = \langle (e_3, e_3) \rangle$, $\mathfrak{m}_l = \langle (le_3, (1+l)e_3), (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (0, \varepsilon e_2) \rangle$,
- 10) $\mathfrak{h}_1 = \langle (e_3, 0), (0, e_3) \rangle$, $\mathfrak{m}_1 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_2) \rangle$,
- 11) $\mathfrak{h}_3 = \langle (e_3, 0), (0, e_1) \rangle$, $\mathfrak{m}_3 = \langle (e_1, 0), (e_2, 0), (0, e_2), (0, e_3) \rangle$,
- 12) $\mathfrak{h}_4 = \langle (e_1, 0), (0, e_1) \rangle$, $\mathfrak{m}_4 = \langle (e_2, 0), (e_3, 0), (0, e_2), (0, e_3) \rangle$,
- 13) $\mathfrak{h}_{13} = \langle (e_1, e_1) \rangle$, $\mathfrak{m}_m = \langle (e_2, 0), (e_3, 0), (0, e_3), (0, e_2), (me_1, (1+m)e_1) \rangle$,
- 14) $\mathfrak{h}_{14} = \langle (e_1, e_2 + e_3) \rangle$, $\mathfrak{m}_n = \langle (e_2, 0), (e_3, 0), (0, e_1), (0, e_2 + e_3), (ne_1, e_2) \rangle$,

where $a, b, c, d, f, g, h, k, l, m, n \in \mathbb{R}$ and $\varepsilon = i$ for $\mathfrak{g}_2 = \mathfrak{so}_3(\mathbb{R})$ whereas $\varepsilon = 1$ for $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{R})$. The cases 1) till 10) occur for both simple 3-dimensional Lie algebras whereas the cases 10) till 14) occur only for $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.

Proof. The basis elements of an arbitrary complement \mathfrak{m}_i to \mathfrak{h}_i , $i = 1, \dots, 18$, in $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2$, where \mathfrak{g}_2 is either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}_3(\mathbb{R})$, are:
In the case $i = 1$

$$(e_1 + a_1e_3, a_2e_3), \quad (e_2 + b_1e_3, b_2e_3), \quad (c_1e_3, e_1 + c_2e_3), \quad (d_1e_3, e_2 + d_2e_3),$$

in the case $i = 2$

$$(e_1 + a_1e_3, a_2(e_2 + e_3)), \quad (e_2 + b_1e_3, b_2(e_2 + e_3)), \\ (c_1e_3, e_1 + c_2(e_2 + e_3)), \quad (d_1e_3, e_3 + d_2(e_2 + e_3)),$$

in the case $i = 3$

$$(e_1 + a_1e_3, a_2e_1), \quad (e_2 + b_1e_3, b_2e_1), \quad (c_1e_3, e_2 + c_2e_1), \quad (d_1e_3, e_3 + d_2e_1),$$

in the case $i = 4$

$$(e_2 + a_1e_1, a_2e_1), \quad (e_3 + b_1e_1, b_2e_1), \quad (c_1e_1, e_2 + c_2e_1), \quad (d_1e_1, e_3 + d_2e_1),$$

in the case $i = 5$

$$(e_2 + a_1e_1, a_2(e_2 + e_3)), \quad (e_3 + b_1e_1, b_2(e_2 + e_3)), \\ (c_1e_1, e_1 + c_2(e_2 + e_3)), \quad (d_1e_1, e_3 + d_2(e_2 + e_3)),$$

in the case $i = 6$

$$(e_1 + a_1(e_2 + e_3), a_2(e_2 + e_3)), \quad (e_3 + b_1(e_2 + e_3), b_2(e_2 + e_3)), \\ (c_1(e_2 + e_3), e_1 + c_2(e_2 + e_3)), \quad (d_1(e_2 + e_3), e_3 + d_2(e_2 + e_3)),$$

in the case $i = 7$

$$(e_3 + a_1e_1 + a_2(e_2 + e_3), 0), \quad (b_1e_1 + b_2(e_2 + e_3), \varepsilon e_1), \\ (c_1e_1 + c_2(e_2 + e_3), \varepsilon e_2), \quad (d_1e_1 + d_2(e_2 + e_3), e_3),$$

in the case $i = 8$

$$(e_1 + a_1e_3, 0), \quad (e_2 + a_2e_3, 0), \quad (a_3e_3, \varepsilon e_1), \quad (a_4e_3, \varepsilon e_2), \quad (a_5e_3, e_3),$$

in the case $i = 9$

$$(e_2 + a_1e_1, 0), \quad (e_3 + a_2e_1, 0), \quad (a_3e_1, \varepsilon e_1), \quad (a_4e_1, \varepsilon e_2), \quad (a_5e_1, e_3),$$

in the case $i = 10$

$$(e_2 + a_1(e_2 + e_3), 0), \quad (e_1 + a_2(e_2 + e_3), 0), \quad (a_3(e_2 + e_3), \varepsilon e_1), \\ (a_4(e_2 + e_3), \varepsilon e_2), \quad (a_5(e_2 + e_3), e_3),$$

in the case $i = 11$

$$(e_3 + a_1e_1 + a_2(e_2 + e_3), a_1e_1 + a_2(e_2 + e_3)), \\ (b_1e_1 + b_2(e_2 + e_3), e_1 + b_1e_1 + b_2(e_2 + e_3)), \\ (c_1e_1 + c_2(e_2 + e_3), e_2 + c_1e_1 + c_2(e_2 + e_3)), \\ (d_1e_1 + d_2(e_2 + e_3), e_3 + d_1e_1 + d_2(e_2 + e_3)),$$

in the case $i = 12$

$$(e_3 + a_1e_1 + a_2(e_2 + e_3), a_1k), \quad (b_1e_1 + b_2(e_2 + e_3), \varepsilon e_1 + b_1k),$$

$$(c_1e_1 + c_2(e_2 + e_3), \varepsilon e_2 + c_1k), (d_1e_1 + d_2(e_2 + e_3), e_3 + d_1k),$$

in the case $i = 13$

$$(e_2 + a_1e_1, a_1e_1), (e_3 + a_2e_1, a_2e_1), (a_3e_1, e_1 + a_3e_1), \\ (a_4e_1, e_2 + a_4e_1), (a_5e_1, e_3 + a_5e_1),$$

in the case $i = 14$

$$(e_2 + a_1e_1, a_1(e_2 + e_3)), (e_3 + a_2e_1, a_2(e_2 + e_3)), (a_3e_1, e_1 + a_3(e_2 + e_3)), \\ (a_4e_1, e_2 + a_4(e_2 + e_3)), (a_5e_1, e_3 + a_5(e_2 + e_3)),$$

in the case $i = 15$

$$(e_2 + a_1(e_2 + e_3), a_1(e_2 + e_3)), (e_1 + a_2(e_2 + e_3), a_2(e_2 + e_3)), \\ (a_3(e_2 + e_3), e_1 + a_3(e_2 + e_3)), (a_4(e_2 + e_3), e_2 + a_4(e_2 + e_3)), \\ (a_5(e_2 + e_3), e_3 + a_5(e_2 + e_3)),$$

in the case $i = 16$

$$(e_2 + a_1e_1, a_1e_3), (e_3 + a_2e_1, a_2e_3), (a_3e_1, \varepsilon e_1 + a_3e_3), \\ (a_4e_1, \varepsilon e_2 + a_4e_3), (a_5e_1, e_3 + a_5e_3),$$

in the case $i = 17$

$$(e_2 + a_1(e_2 + e_3), a_1e_3), (e_1 + a_2(e_2 + e_3), a_2e_3), (a_3(e_2 + e_3), \varepsilon e_1 + a_3e_3), \\ (a_4(e_2 + e_3), \varepsilon e_2 + a_4e_3), (a_5(e_2 + e_3), e_3 + a_5e_3),$$

in the case $i = 18$

$$(e_1 + a_1e_3, a_1e_3), (e_2 + a_2e_3, a_2e_3), (a_3e_3, \varepsilon e_1 + a_3e_3), \\ (a_4e_3, \varepsilon e_2 + a_4e_3), (a_5e_3, e_3 + a_5e_3),$$

where $a_i, i = 1, 2, \dots, 5, b_j, j = 1, 2, c_k, k = 1, 2, d_l, l = 1, 2,$ are real parameters, $\varepsilon = i$ for $\mathfrak{g}_2 = \mathfrak{so}_3(\mathbb{R})$ and $\varepsilon = 1$ for $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.

Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i, i = 1, \dots, 18,$ and Lemma 17 we obtain the assertion. \square

Proposition 21. *Let G be locally isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where G_2 is either $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. If G is the group topologically generated by the left translations of a connected almost differentiable left A -loop L then L is either a Scheerer extension of G_2 by \mathbb{H}_2 or the direct product $\mathbb{H}_2 \times \mathbb{H}_2$, where \mathbb{H}_2 denotes the hyperbolic plane loop. In the second case G is isomorphic to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.*

Proof. Since we assume that $\dim L \geq 4$ we have to consider only the pairs (\mathbf{h}, \mathbf{m}) in Proposition 20. Now using **1.1** and **1.2** we obtain that the element $(0, e_1) \in \mathbf{h}_3 \cap \mathbf{h}_4$, the element $(e_1, 0) \in \mathbf{h}_9$, the element $(e_1, e_1) \in \mathbf{h}_{13}$ respectively the element $(e_1, e_2 + e_3) \in \mathbf{h}_{14}$ is conjugate in this order to

$(0, e_2) \in \mathbf{m}_3 \cap \mathbf{m}_4$, to $(e_2, 0) \in \mathbf{m}_d \cap \mathbf{m}_f \cap \mathbf{m}_g$, to $(e_2, e_2) \in \mathbf{m}_m$ respectively to $(e_2, e_2 + e_3) \in \mathbf{m}_n$. Hence there exists no global left A-loop L such that $T_e L$ is a reductive complement listed in the cases 4), 5), 6), 11), 12), 13), 14) (see. Lemma 3).

Now we consider the reductive complements $\mathbf{m}_a, \mathbf{m}_b, \mathbf{m}_c$ in 1) till 3) of Proposition 20. First we assume that $a \neq 0, b \neq 0, c \neq 0$. The vectors $v_{j,l} = (ke_3, \frac{k}{l}\varepsilon e_j)$, $w_{j,l} = (\sqrt{k^2 - 4\pi^2}e_2 + ke_3, \frac{k}{l}\varepsilon e_j)$, where $k > 2\pi$ is an integer, are contained in the subspace \mathbf{m}_a for $j = 3, l = a$ and $\varepsilon = 1$, in the subspace \mathbf{m}_b for $j = 2, l = b$, respectively in \mathbf{m}_c for $j = 1, l = c$, where $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$ and $\varepsilon = i$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$. According to 1.1 and 1.2 the images of $v_{j,l}, w_{j,l}$, $j = 1, 2, 3$, under the exponential map have the following representatives in $PSL_2(\mathbb{R}) \times G_2$:

$$\begin{aligned} m_1 &= \exp v_{3,a} = \left(A, \begin{pmatrix} \cos \frac{k}{a} & \sin \frac{k}{a} \\ -\sin \frac{k}{a} & \cos \frac{k}{a} \end{pmatrix} \right), \\ m_2 &= \exp w_{3,a} = \left(I, \begin{pmatrix} \cos \frac{k}{a} & \sin \frac{k}{a} \\ -\sin \frac{k}{a} & \cos \frac{k}{a} \end{pmatrix} \right), \\ m_3 &= \exp v_{2,b} = \left(A, \begin{pmatrix} \cosh(\frac{k}{b}\varepsilon) & \sinh(\frac{k}{b}\varepsilon) \\ -\sinh(\frac{k}{b}\varepsilon) & \cosh(\frac{k}{b}\varepsilon) \end{pmatrix} \right), \\ m_4 &= \exp w_{2,b} = \left(\pm I, \begin{pmatrix} \cosh(\frac{k}{b}\varepsilon) & \sinh(\frac{k}{b}\varepsilon) \\ -\sinh(\frac{k}{b}\varepsilon) & \cosh(\frac{k}{b}\varepsilon) \end{pmatrix} \right), \\ m_5 &= \exp v_{1,c} = \left(A, \begin{pmatrix} \cosh(\frac{k}{c}\varepsilon) + \sinh(\frac{k}{c}\varepsilon) & 0 \\ 0 & \cosh(\frac{k}{c}\varepsilon) - \sinh(\frac{k}{c}\varepsilon) \end{pmatrix} \right), \\ m_6 &= \exp w_{1,c} = \left(I, \begin{pmatrix} \cosh(\frac{k}{c}\varepsilon) + \sinh(\frac{k}{c}\varepsilon) & 0 \\ 0 & \cosh(\frac{k}{c}\varepsilon) - \sinh(\frac{k}{c}\varepsilon) \end{pmatrix} \right), \end{aligned}$$

where $A = \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix}$, $\varepsilon = i$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$, whereas $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$. For the representatives

$$\begin{aligned} g_1 &= \left(I, \begin{pmatrix} \cos \frac{k}{a} & \sin \frac{k}{a} \\ -\sin \frac{k}{a} & \cos \frac{k}{a} \end{pmatrix} \right), \\ g_2 &= \left(I, \begin{pmatrix} \cosh(\frac{k}{b}\varepsilon) & \sinh(\frac{k}{b}\varepsilon) \\ -\sinh(\frac{k}{b}\varepsilon) & \cosh(\frac{k}{b}\varepsilon) \end{pmatrix} \right), \\ g_3 &= \left(I, \begin{pmatrix} \cosh(\frac{k}{c}\varepsilon) + \sinh(\frac{k}{c}\varepsilon) & 0 \\ 0 & \cosh(\frac{k}{c}\varepsilon) - \sinh(\frac{k}{c}\varepsilon) \end{pmatrix} \right) \end{aligned}$$

we have $g_1 = m_1 \cdot h_1 = m_2$, $g_2 = m_3 \cdot h_1 = m_4$, $g_3 = m_5 \cdot h_1 = m_6$ such that $h_1 = (A^{-1}, I)$. These facts again contradict Lemma 3.

For $a = 0, b = 0, c = 0$ the complements $\mathbf{m}_a, \mathbf{m}_b, \mathbf{m}_c$ in 1) till 3) of Proposition 20 reduce to $\mathbf{m}_0 = \langle (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (0, e_3) \rangle$. The

exponential image $\exp \mathbf{m}_0$ is the direct product $M \times G_2$, such that M is the image of the section corresponding to the hyperbolic plane loop \mathbb{H}_2 (cf. [18], pp. 283-284) and G_2 is the group $PSL_2(\mathbb{R})$ respectively $SO_3(\mathbb{R})$ according whether $\varepsilon = 1$ or $\varepsilon = i$. Since H has the shape $H_1 \times \{1\}$, where $H_1 \cong SO_2(\mathbb{R}) \leq PSL_2(\mathbb{R})$ the global loop L_0 realized on $\exp \mathbf{m}_0$ is the direct product of \mathbb{H}_2 and G_2 .

Now we treat the complements $\mathbf{m}_h, \mathbf{m}_k, \mathbf{m}_l, h, k, l \in \mathbb{R}$ of the cases 7) till 9) in Proposition 20. The reductive complement $\mathbf{m}_a, a \in \mathbb{R}, \mathbf{m}_b, b \in \mathbb{R}$, respectively $\mathbf{m}_c, c \in \mathbb{R}$ of Lemma 12 in [6], p. 404, is in this order a subspace of $\mathbf{m}_h, \mathbf{m}_k$, respectively \mathbf{m}_l . Moreover, the subalgebra \mathbf{h}_{16} in the case 7) coincides with the subalgebra \mathbf{h} in case 1) of Lemma 12 in [6], the subalgebra \mathbf{h}_{17} in the case 8) is equal with the subalgebra \mathbf{h} in case 2) of Lemma 12 in [6], and the subalgebra \mathbf{h}_{18} in the case 9) coincides with the subalgebra \mathbf{h} in case 3) of Lemma 12 in [6], p. 404. Hence the same computations as in the proof of Proposition 13 in [6], pp. 404-406, show that for $h \neq -1$ the complement \mathbf{m}_h , for $k \neq 0$ the complement \mathbf{m}_k and for $l \notin \{0, -1\}$ the complement \mathbf{m}_l cannot be the tangent space of a global almost differentiable left A-loop.

It remains to consider the complements $\mathbf{m}_{h=-1}, \mathbf{m}_{k=0}, \mathbf{m}_{l=0}$ and $\mathbf{m}_{l=-1}$. First let $\varepsilon = i$. Then the element $(e_1, e_3) \in \mathbf{h}_{16}$ is conjugate to $(e_2, ie_1) \in \mathbf{m}_{h=-1}$, the element $(e_2 + e_3, e_3) \in \mathbf{h}_{17}$ is conjugate to $(e_2 + e_3, ie_1) \in \mathbf{m}_{k=0}$ and the element $(e_3, e_3) \in \mathbf{h}_{18}$ is conjugate to $(e_3, ie_1) \in \mathbf{m}_{l=-1}$ (see 1.2), which are contradictions to Lemma 3. Since the exponential image of the Lie algebra \mathbf{h}_{18} has the shape $H_n = \{(x, x^n) \mid x \in SO_2(\mathbb{R}), n \in \mathbb{N} \setminus \{0\}\}$ the exponential image $M \times SO_3(\mathbb{R})$ of the complement $\mathbf{m}_{l=0}$, where M is the image of the section belonging to the hyperbolic plane loop \mathbb{H}_2 (cf. [18], pp. 283-284), yields Scheerer extensions of $SO_3(\mathbb{R})$ by \mathbb{H}_2 (cf. [18], Section 2).

Finally let $\varepsilon = 1$. The complements $\mathbf{m}_{h=-1}, \mathbf{m}_{k=0}, \mathbf{m}_{l=-1}$ and $\mathbf{m}_{l=0}$ are (up to interchanging the components) equal to the vector space

$$\mathbf{m}' = \langle (e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2) \rangle$$

and its exponential image $\exp \mathbf{m}'$ is the direct product $PSL_2(\mathbb{R}) \times M$, where M is the image of the section corresponding to \mathbb{H}_2 . The group $H = \{(\varphi(x), x) \mid x \in SO_2(\mathbb{R})\}$ coincides with the group H_{16} belonging to \mathbf{h}_{16} respectively with H_{17} of \mathbf{h}_{17} if φ is a homomorphism from $SO_2(\mathbb{R})$ onto a hyperbolic respectively a parabolic 1-parameter subgroup of $PSL_2(\mathbb{R})$. The subgroup H_{18} of \mathbf{h}_{18} has the form: $H'_n = \{(x^n, x) \mid x \in SO_2(\mathbb{R}), n \in \mathbb{N} \setminus \{0\}\}$. According to [18], Section 2, any loop L realized on the factor space G/H_n , $n = 16, 17, 18$, and having $\exp \mathbf{m}'$ as the image of its section is a Scheerer extension of the Lie group $PSL_2(\mathbb{R})$ by \mathbb{H}_2 .

All Scheerer extensions having $PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$ or $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ as the group topologically generated by their left translations satisfy the Bol identity because of $[[\mathbf{m}, \mathbf{m}], \mathbf{m}] \subset \mathbf{m}$ but they are not Bruck loops since

there is no involutory automorphism $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\sigma(\mathfrak{m}) = -\mathfrak{m}$ and $\sigma(\mathfrak{h}) = \mathfrak{h}$.

In the remaining case 10) in Proposition 20 the subgroup H_1 of \mathfrak{h}_1 is the direct product $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and the exponential image M_1 of \mathfrak{m}_1 is the direct product $M \times M$, where M is the image of the section belonging to \mathbb{H}_2 . According to Proposition 1.19 in [18], p. 28, the loop L is the direct product $\mathbb{H}_2 \times \mathbb{H}_2$. \square

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